

**Variable Annuity Guarantees The GMWB**  
**IFID Centre's 4th Annual Conference**

**Tuesday, May 31st , 2005**

**Phelim Boyle<sup>◇</sup>, Thangaraj Draviam<sup>◇</sup> and Yan Liu<sup>◇</sup>**

<sup>◇</sup>University of Waterloo

## Acknowledgements

Thanks to Deep Patel for helpful comments

# Outline

- Background on Variable Annuities
- Investor objectives
- The **Guaranteed Minimum Withdrawal Benefit(GMWB)**
- Basic features
- Optimal investment strategy for fixed consumption stream
- Pricing in an ideal setting
- Sensitivity to financial variables
- Actuarial variables
- Modeling: Multiple state models
- Dynamic behavior
- Concluding remarks

## Background

- Baby boomers now getting close to retirement
- Great interest in annuities
- Investors like to have possibility of upside appreciation
- Investors also concerned about downside risk
- Proliferation of insurance and annuity products that combine both features
- Equity Indexed Annuities and Variable Annuities(VA)
- Variable annuities combine different embedded options
- The latest type of product is called **Guaranteed Minimum Withdrawal Benefit(GMWB)**
- Very popular in the market
- Milevsky and Salisbury (2004)

## GMWB

- The GMWB adds a guaranteed floor of withdrawal benefits to a VA
- It provides a guaranteed level of income to the policyholder.
- For example suppose investor puts \$100,000 in a VA. Assume this is invested in a stock(balanced) portfolio
- Policyholder can withdraw a certain fixed percentage (7% is typical) every year until the initial premium is withdrawn
- Assume the withdrawal rate is 7% per annum.
- Our policyholder could withdraw \$7,000 each year until the total withdrawals reach \$100,000. This takes 14.28 years.
- Note the policyholder can withdraw the funds **irrespective of how the investment account performs**
- Here is an example. Market does well at first and then collapses.

Year	Rate on fund	Fund before withdrawal	Fund after withdrawal	Amount withdrawn	Balance remaining
1	10%	110,000	103,000	7,000	93,000
2	10%	113,300	106,300	7,000	86,000
3	-60%	42,520	35,520	7,000	79,000
4	-60%	14,208	7,208	7,000	72,000
5	- 2.8857%	7,000	Zero	7,000	65,000
6	r%	0	0	7,000	58,000
⋮	⋮	⋮	⋮	⋮	⋮
14	r%	0	0	7,000	2000

## Portfolio problem

- Recall Merton's approach to optimal investment consumption
- Risky asset and risk free asset.
- Investor maximizes expected utility of consumption
- Under some assumptions<sup>1</sup> Merton found optimal solution
- Solution very intuitive
- Constant proportion invested in stock

$$\pi = \frac{\mu - r}{\sigma^2(1 - \alpha)}$$

where  $\mu$  is expected stock return,  $r$  is riskfree rate  $\sigma^2$  is variance of stock return and  $1 - \alpha$  is the relative risk aversion.

---

<sup>1</sup>Power utility, lognormal asset returns, constant risk free rate .

## Guaranteed consumption level

- Suppose investor desires a fixed consumption level  $C_1 > 0$ .
- Solved by Karatzas and Shreve
- Optimal investment is

$$\pi(w(t), t) = \frac{\mu - r}{\sigma^2(1 - \alpha)} \frac{w(t) - l(t)}{w(t)}, \quad w(t) \geq l(t).$$

where  $w(t)$  is current wealth and

$$l(t) = \frac{C_1}{r} \{1 - e^{-r(T-t)}\}.$$

- As wealth gets closer to guaranteed amount investor shifts towards the riskfree asset. Strategy is **hard to implement. Lower allocation to stock.**

## Example

- Assume

$$\mu = .08, \quad r = .04, \quad \sigma = .2, \quad \alpha = -1$$

then Merton ratio is 50% in stocks

- This fraction does not depend on investor's wealth.
- Suppose that we have a 15 year horizon and that

$$C_1 = \frac{100}{15}$$

- We simulate one stock price path and find the fraction invested in stock.
- If the investors wealth approaches the lower bound,  $l(t)$  most of the funds will be in the risk asset and the portfolio will earn the risk free rate.

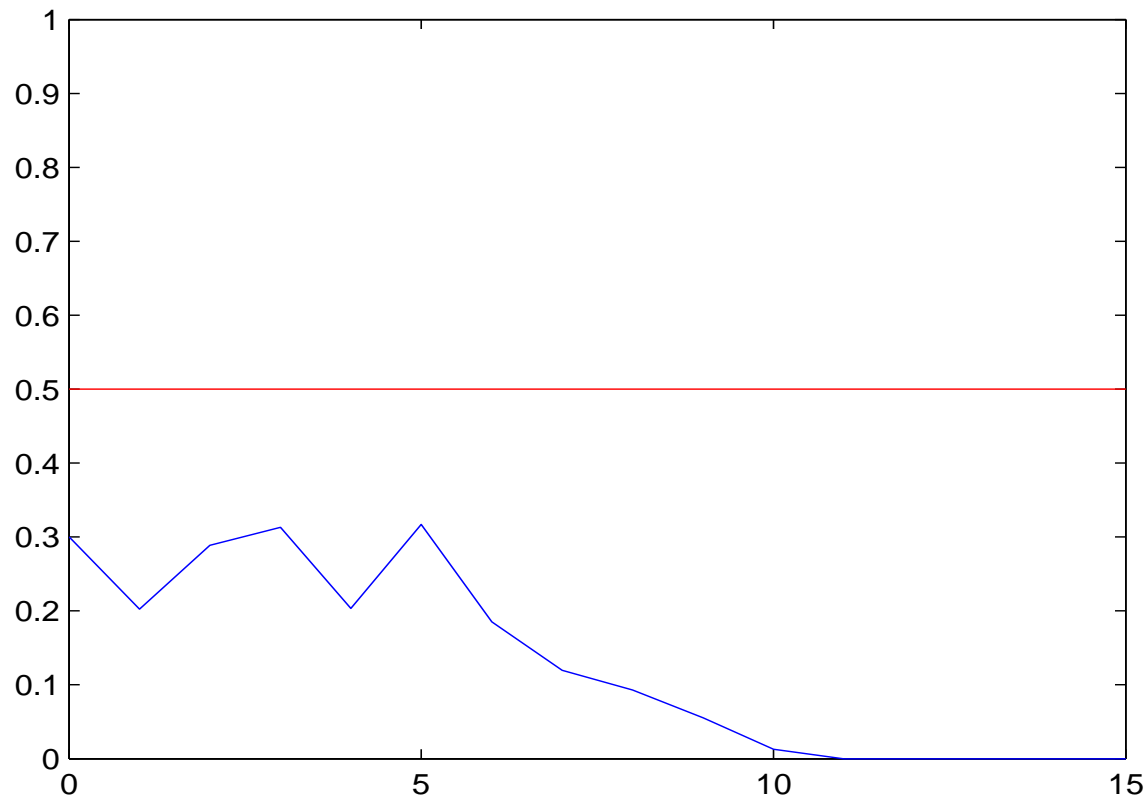


Figure 1: red line is fraction invested in stock in the unconstrained Merton problem. Ratio = 0.5; Corresponding fraction when there is a guaranteed floor. Blue line results based on one simulation.

## The GMWB

- The GMWB rider adds this benefit.
- Policyholder can guarantee a fixed level of income no matter what happens to the market
- Fee for the GMWB is expressed as a percentage (say fifty basis points) of either
  - The investment account **or**
  - The outstanding guaranteed withdrawal benefit.

## Assumptions

- Perfect frictionless market
- Ignore lapses, partial withdrawals mortality etc.
- Assume max amount taken each year
- Fixed term contract over  $[0, T]$ .
- Pure investment fund dynamics

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where  $B_t$  is a Brownian motion under  $P$

$\mu$  is the drift

$\sigma$  is the volatility.

## Investor's account

- Let  $V_t$  be the value of the investors account at time  $t$ .
- $V$  has an **absorbing barrier at zero**. Suppose first time it hits zero is  $\tau$ .
- Dynamics of  $V$  for  $0 < t < \tau$  are

$$dV_t = [(\mu - q)V_t - g]dt + \sigma V_t dB_t$$

where  $q$  is the fee and  $g$  is the withdrawal rate.

- If the initial investment amount is  $I_0$  then

$$g = \frac{I_0}{T}.$$

## Pricing the contract

There are two ways to decompose the GMWB.

### Call Option decomposition

Investor pays an initial amount  $I_0$  at time zero. Benefit is a guaranteed stream of  $g$  per annum plus a call option on the terminal amount  $V_T$ . Strike price of the call is zero.  $q$  is the control variable. Find  $q$  so that

$$I_0 = \int_0^T g e^{-ru} du + C_0$$

where  $r$  is the (constant) risk free rate and  $C_0$  is the time zero value of the call. Fixed point ideas. Solve using pde or Monte Carlo. We used both: MC with variance reduction

# Pricing the contract

## Put Option decomposition

- Investment often in mutual fund.
- Insurer guarantees to pay remaining withdrawal benefits if  $V(t)$  reaches zero in  $[0, T]$ .
- Guarantee provided by the insurance is a **put option**.
- If policyholder's investment account stays positive there is no payment under the put option.
- The option is exercised automatically when the account balance first becomes zero.

## Put Option decomposition

- When this happens the insurer agrees to pay the remaining stream of withdrawal benefits of  $g$  per annum.

$$\int_{\tau}^T g e^{-ru} du$$

- Put option has a random exercise time  $\tau$ . Put is funded by the fee payable until time  $\tau$ .
- Guarantees backed solely by the claims paying ability of XYZ insurance co.

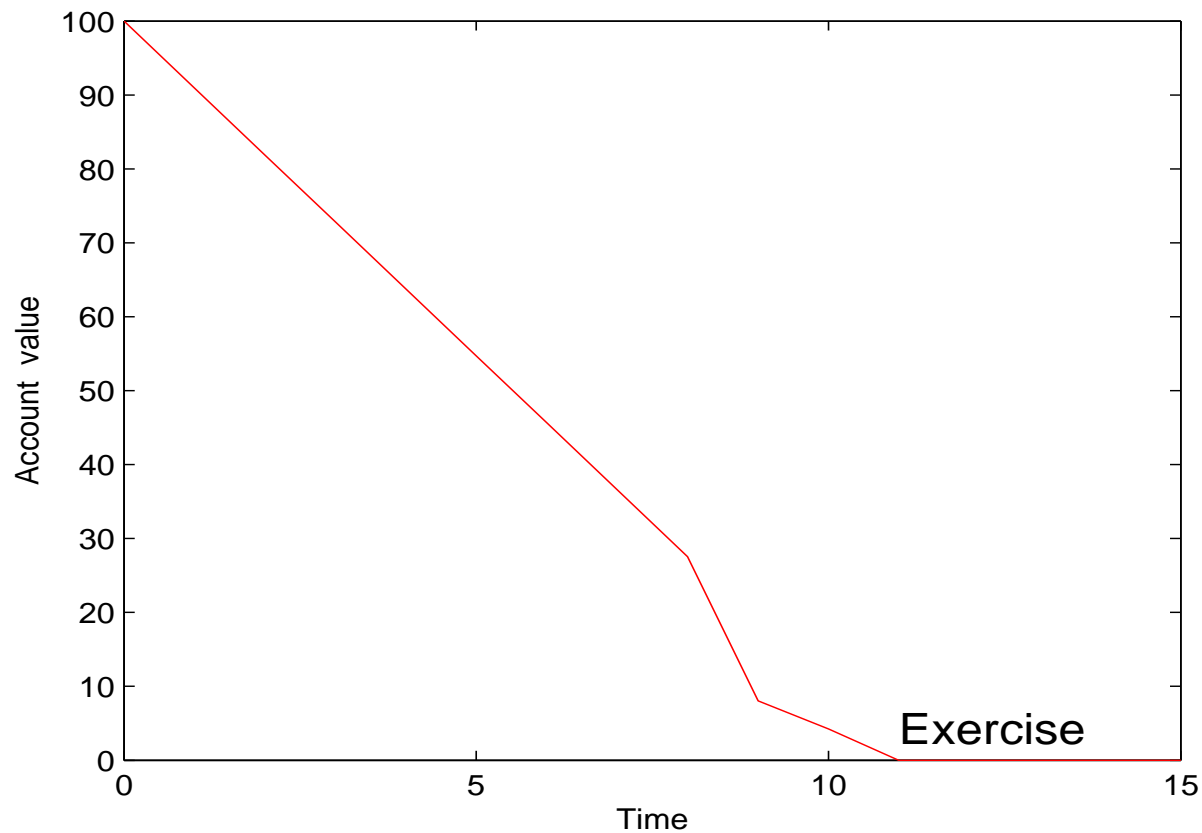


Figure 2: Path of  $V$  reaching zero at year 11. Put option exercised when  $V$  hits zero. Value of put then is  $\int_{\tau}^T g e^{-ru} du$

## Idealized Contract: Numerical Example

Benchmark contract, fifteen year term. No lapses no deaths. All policyholders start to withdraw funds at max rate from the outset. Input parameters are

Parameter	Symbol	Benchmark value
Initial investment	$I_0$	100
Contract term	$T$	15 years
Withdrawal rate	$g$	6.6667
Volatility	$\sigma$	0.20
Riskfree rate	$r$	0.05

Now compute call and put prices for different values of  $q$ .

## Call and put values for benchmark GMWB

Value of $q$ basis points	Present value of Contributions	Put option
0	0	3.98
10	0.96	4.07
25	2.36	4.20
48	4.40	4.40
75	6.79	4.67
100	8.87	4.91
200	16.33	5.98
0300	22.63	7.17

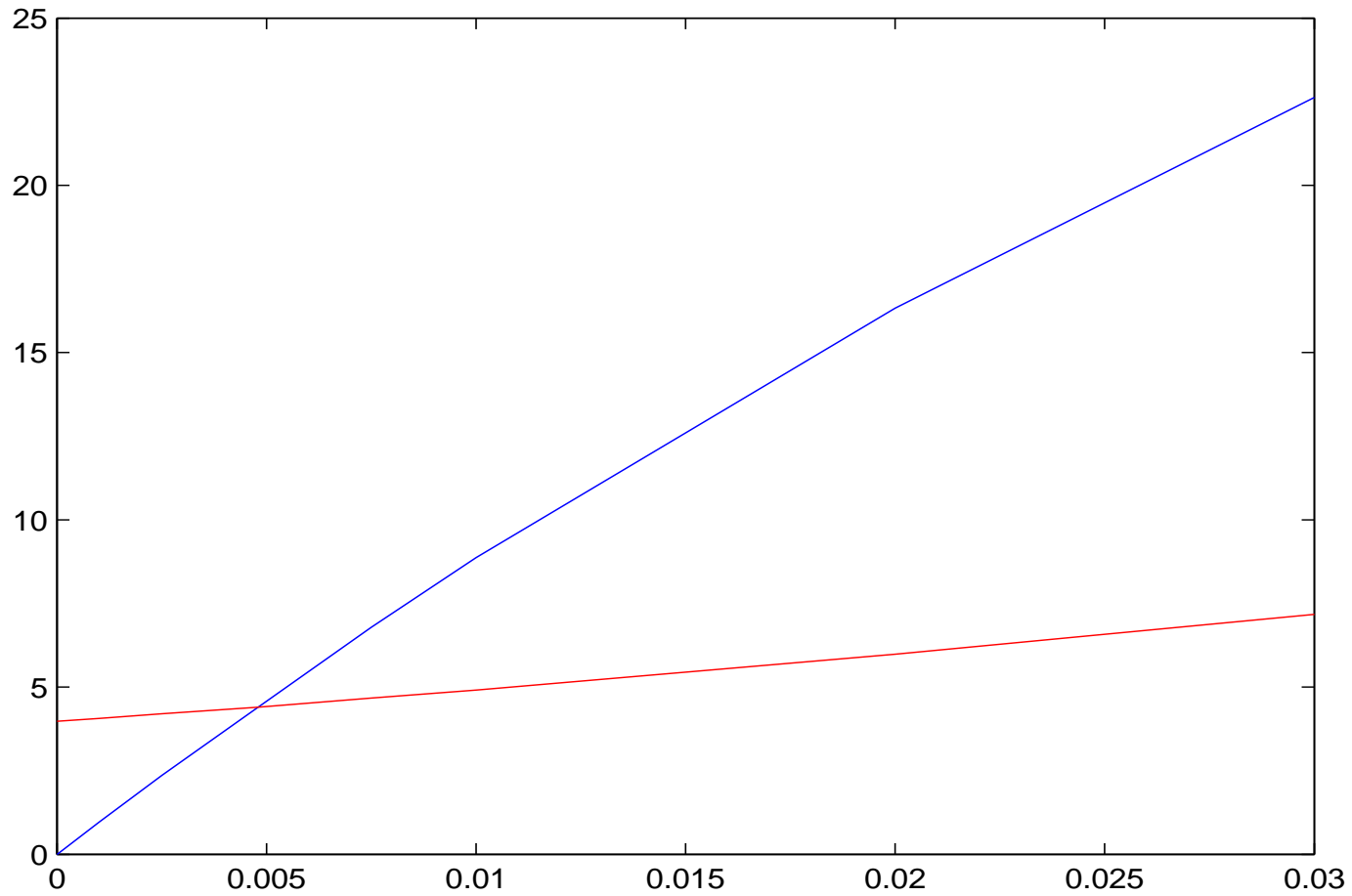


Figure 3: Plot of present value of contributions against value of put option

## No arbitrage Values

For this example the no arbitrage Value of  $q$  is

$$q = 0.004751$$

Call and Put values when  $q = 0.004751$

Entity	Value (sd)
Value of contributions	4.4012(0.0003)
Value of put option	4.4014(0.0008)

## Sensitivity to interest rate

Suppose we have benchmark contract. Fix  $q = 0.004751$ . Explore sensitivity of **put value** and **present value of contributions** to inputs. First vary the interest rate

Interest Assumption	Present Value of Contributions	Put Option
0.04	4.29	6.14
0.05	4.40	4.40
0.06	4.50	3.09

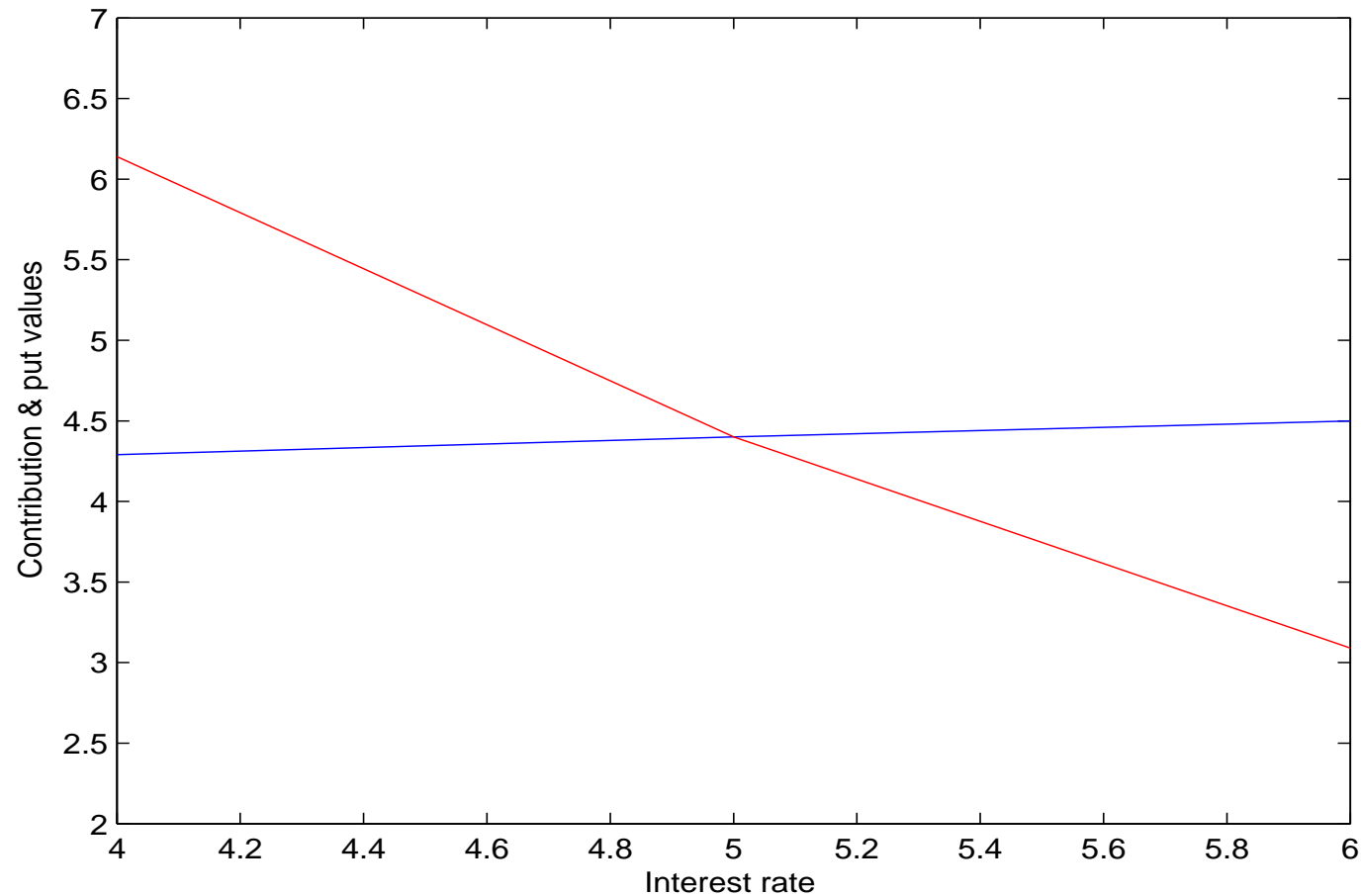


Figure 4: Sensitivity to interest rates. Present value of **contributions** and value of **put option**

## Sensitivity to volatility

Now vary the volatility

Volatility Assumption	Present Value of Contributions	Put Option
0.15	4.37	2.08
0.20	4.40	4.40
0.25	4.44	7.07

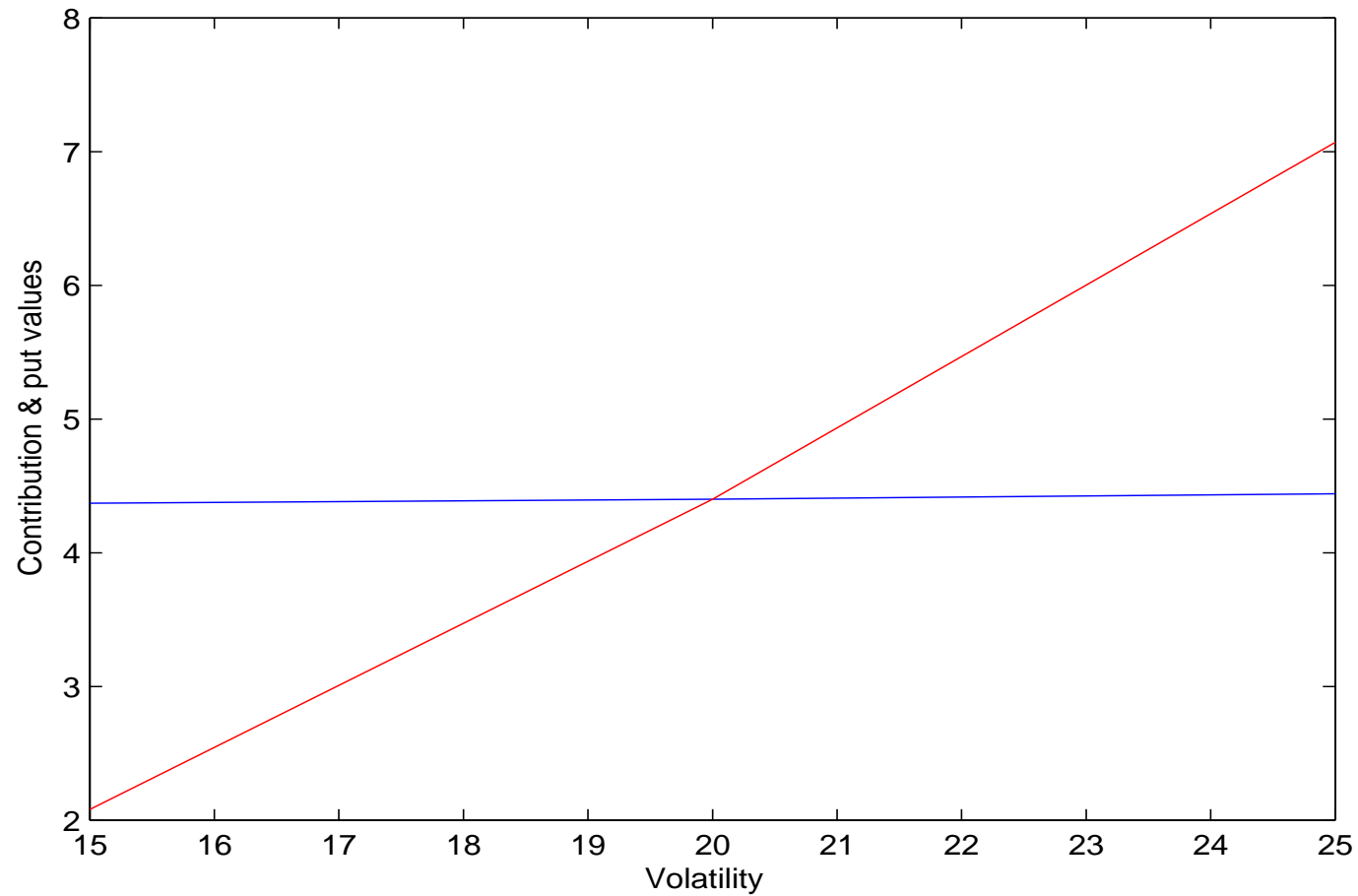


Figure 5: Sensitivity to volatility. Present value of **contributions** and value of **put option**

## Sensitivity to Account Value

Vary  $V_0$  and keep  $g$  fixed at

$$\frac{100}{15} = 6.6667$$

$V_0$ Assumption	Present Value of Contributions	Put Option
------------------	--------------------------------	------------

Conclusion: Moneyness matters

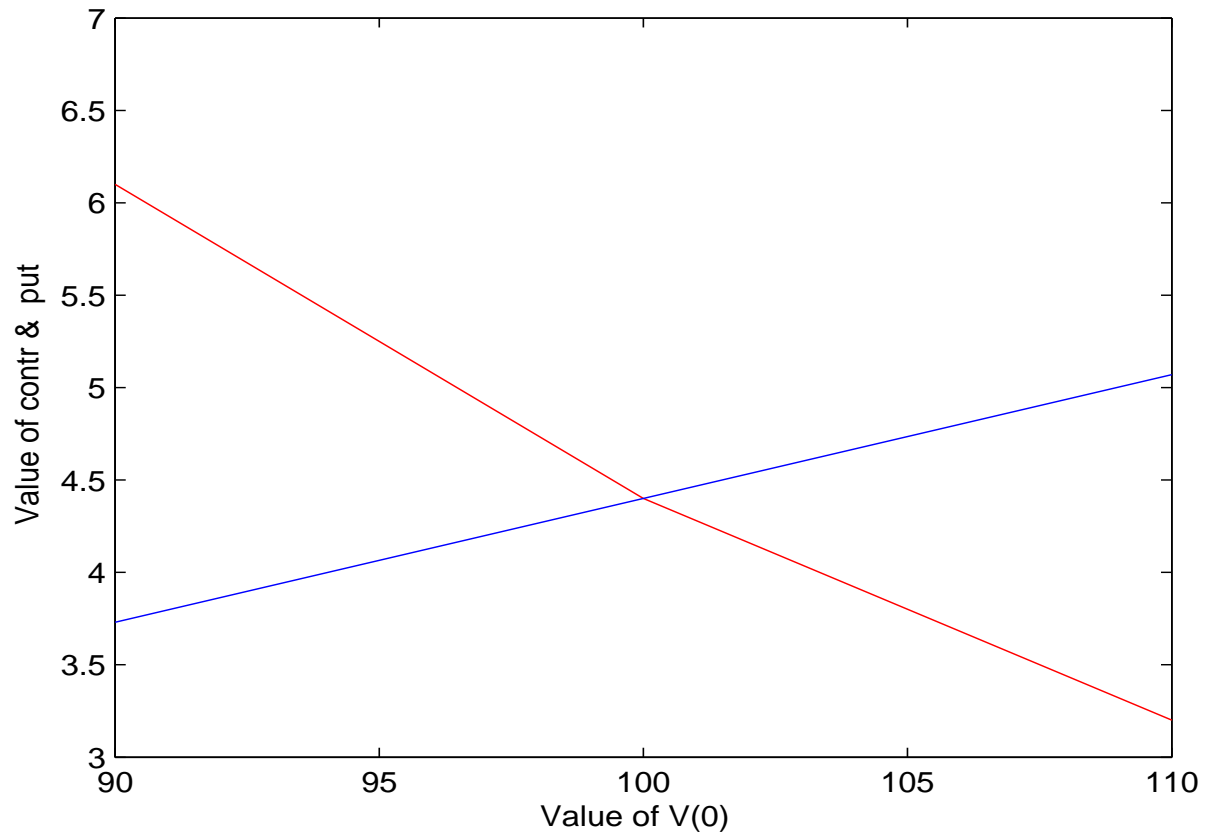


Figure 6: Sensitivity to  $V_0$  with fixed  $g$ . Present value of **contributions** and value of **put option**

## Comments

The put option is much more sensitive to the inputs than the contributions. US insurance companies that write GMWB business have now more incentives to hedge

Hedging can be done in-house or using external agencies, such as reinsurance companies, investment banks or specialized boutiques. Once the product is priced the risk management is carried out on an ongoing basis. Hedging is now **more attractive** because of new accounting regulations. Under SFAS 133 guarantee is classified as a derivative and hedging portfolio can be marked to market.

## Relaxing the assumptions

In practice market is incomplete. Need to consider

- Lapses
- Mortality
- Rate of utilization

Some of these variables interact with realized investment returns.

## Jon Boscia, Chairman and CEO of Lincoln Financial Group

### Quote from Feb 2004 webcast

*GMWB accounted for 40% of total sales. Our GMWB rider gained a strong foothold in key distribution channels as a result of strong sales,” Boscia said. He continued, “Only 7% electing the rider are taking withdrawal at the maximum rate allowed with the guarantee. This level of utilization is very good because it is an indication that we are reaching the target group we aimed for when designing this rider - the comfort buyer still in accumulation stages of life.*

## Discussion

We cannot assume that surrenders follow the pattern of mutual funds or even other VA policies. We expect surrender behaviour to be influenced by the value of guarantee. If the investment account is low a policyholder contemplating surrender now has a stronger incentive to maintain the policy because of the guarantee. This should be factored into the construction of the hedge. It is not easy to do this because:

1. We do not have good theories on exactly how policyholders will behave
2. No published data on actual lapse rates (in particular, dynamic behaviour) for this product
3. The product is a very new one.

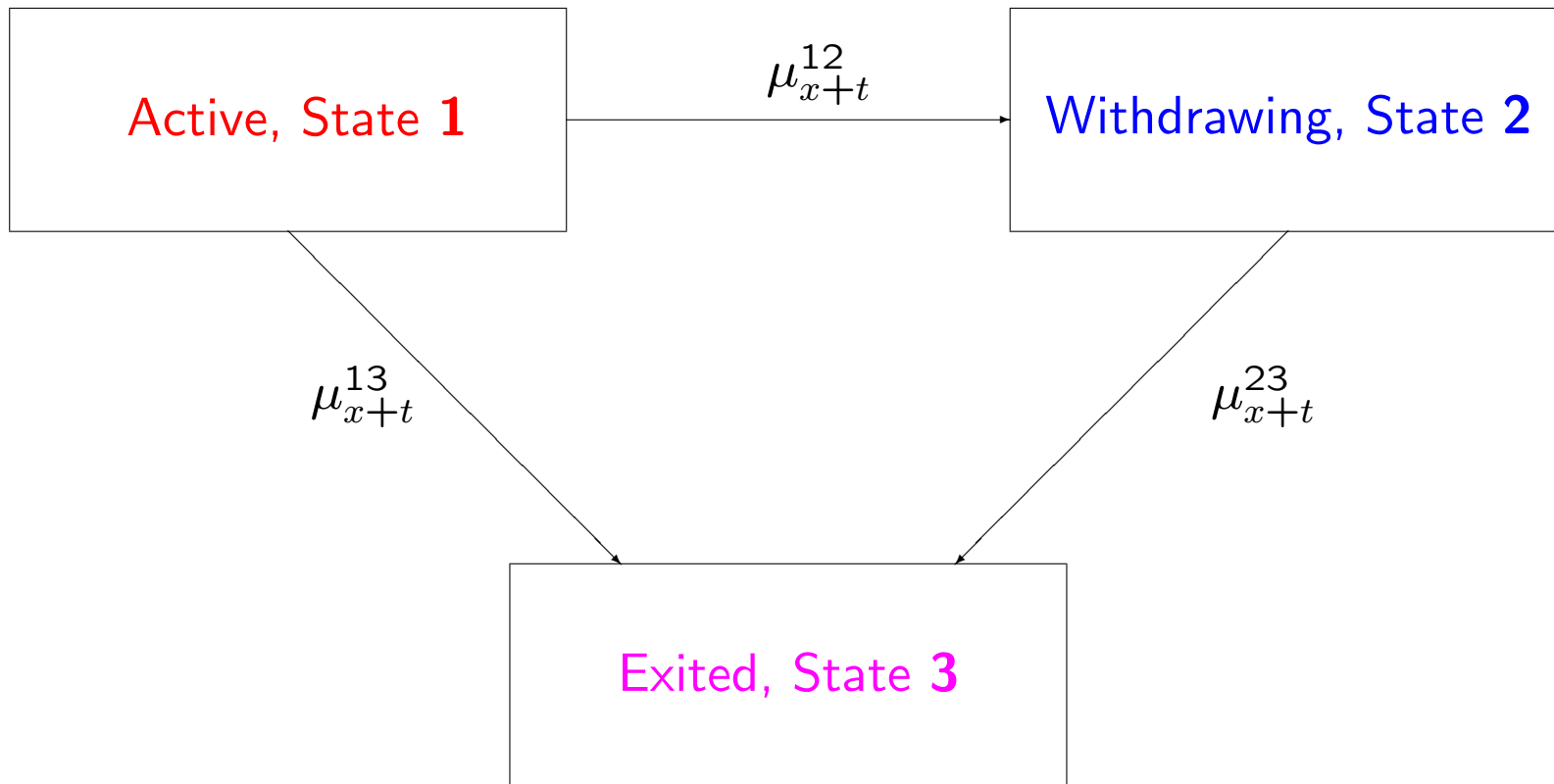
## Multiple State Models

We propose a Multiple State framework to model the situation. We assume there are three states

- In force actives not yet using the withdrawal feature
- Policyholders who are withdrawing under the GMWB.
- Policyholders who have exited (died or lapsed).

We model transition intensities from one state to another. These transition intensities will initially be deterministic functions of time. Later they will depend on economic covariates.

Assume a life enters system at age  $x$ . Let  $\mu_{x+t}^{ij}$  be transition intensity from state  $i$  to state  $j$  at time  $t$  later.



## Transition Intensities

$$\begin{bmatrix} 0 & \mu_{x+t}^{12} & \mu_{x+t}^{13} \\ 0 & 0 & \mu_{x+t}^{23} \\ 0 & 0 & 0 \end{bmatrix}$$

We assume  $\mu_{x+t}^{12}$  is constant (20% baseline assumption)

Surrenders tend to peak when surrender charges drop off .

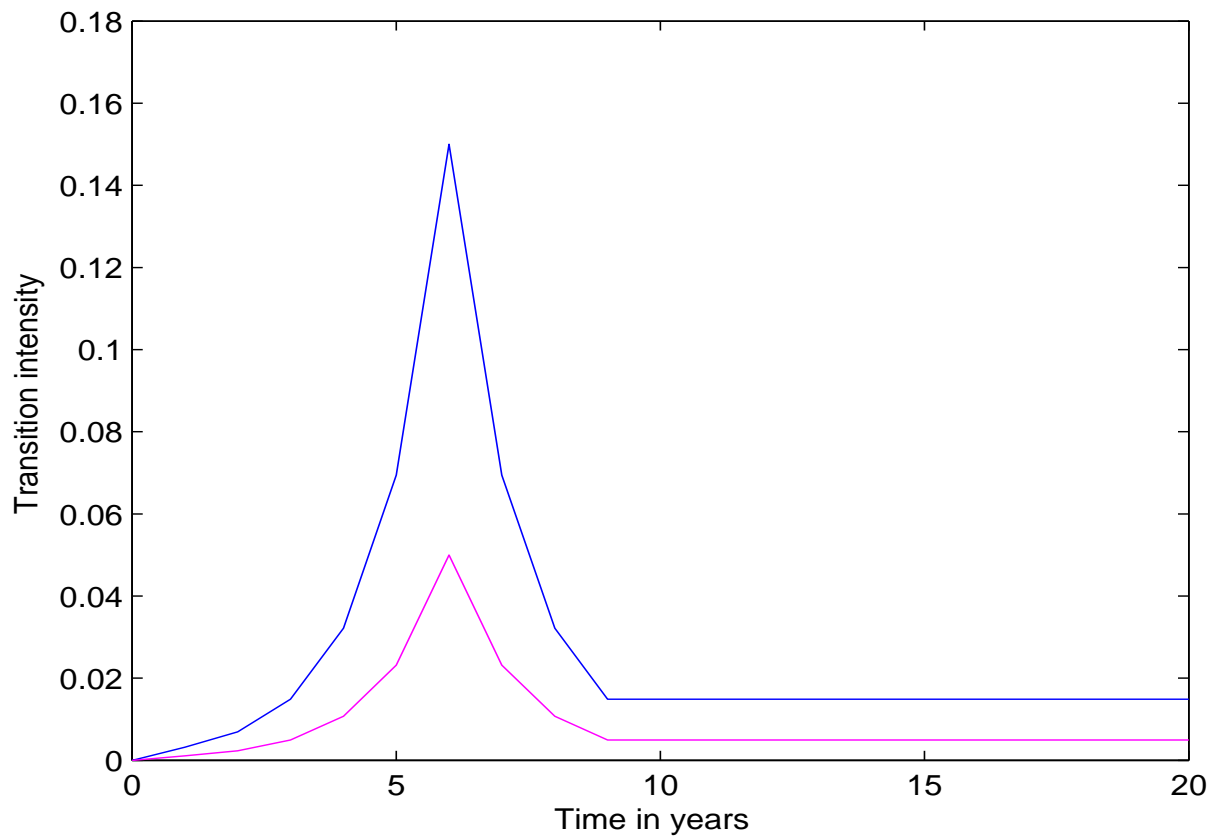


Figure 7: Graph of  $\mu_{x+t}^{13}$  and  $\mu_{x+t}^{23}$

## Transition probability matrix

Assume that transition intensities are piecewise linear(constant) over range  $[t, t + 1]$ . We can compute the transition probability matrix on this basis. Thus for  $t = 0$  this matrix is

$$\begin{pmatrix} e^{-(\mu_x^{12} + \mu_x^{13})} & \frac{\mu_x^{12}}{\mu_x^{12} + \mu_x^{13}} (1 - e^{-(\mu_x^{12} + \mu_x^{13})}) & \frac{\mu_x^{13}}{\mu_x^{12} + \mu_x^{13}} (1 - e^{-(\mu_x^{12} + \mu_x^{13})}) \\ 0 & e^{-\mu_x^{23}} & (1 - e^{-\mu_x^{23}}) \\ 0 & 0 & 1 \end{pmatrix}$$

## Transition probability matrix

We now value the benchmark contract incorporating deterministic intensities. Contract details and assumptions

Parameter	Symbol	Benchmark value
Initial investment	$I_0$	100
Contract term	T	15 years
Withdrawal rate	$g$	6.6667
Volatility	$\sigma$	0.20
Riskfree rate	$r$	0.05

Assume that once a policyholder starts to withdraw they last 15 years unless terminated(lapse or death.)

## Value of put option and contributions

Here are the results. First we assume same fee we found earlier. Hence fee is 47.51 basis points.

Assumption	Value of contributions 47.51 basis points	Value of Put Option
Deterministic intensities	5.54	2.93
Zero lapses. All start to withdraw at outset.	4.40	4.40

Note value of **contributions** increases and value of **put** goes down.

Another way to examine this would be to find the no arbitrage fee that equates the **put value** and the value of the **fee**. This no arbitrage fee is 24 basis points. With this fee the value of both contributions and the put is 2.78.

## Dynamic behavior

- We expect policyholders behaviour to respond to economic conditions.
- If the option is in the money then it is more likely they will start withdrawal rather than surrender.

To capture this assume that  $\bar{\mu}_{x+t}^{12}$ ,  $\bar{\mu}_{x+t}^{13}$  denotes the deterministic intensity function. Then

$$\mu_{x+t}^{12} = \bar{\mu}_{x+t}^{12} e^{\lambda \max\left(0, 1 - \frac{A(t)}{G(t)}\right)}$$

and

$$\mu_{x+t}^{13} = \bar{\mu}_{x+t}^{13} e^{-\lambda \max\left(0, 1 - \frac{A(t)}{G(t)}\right)}$$

where  $A(t)$  is policyholder's account at time  $t$  and  $G(t)$  is outstanding balance on the total withdrawal amount at time  $t$ .

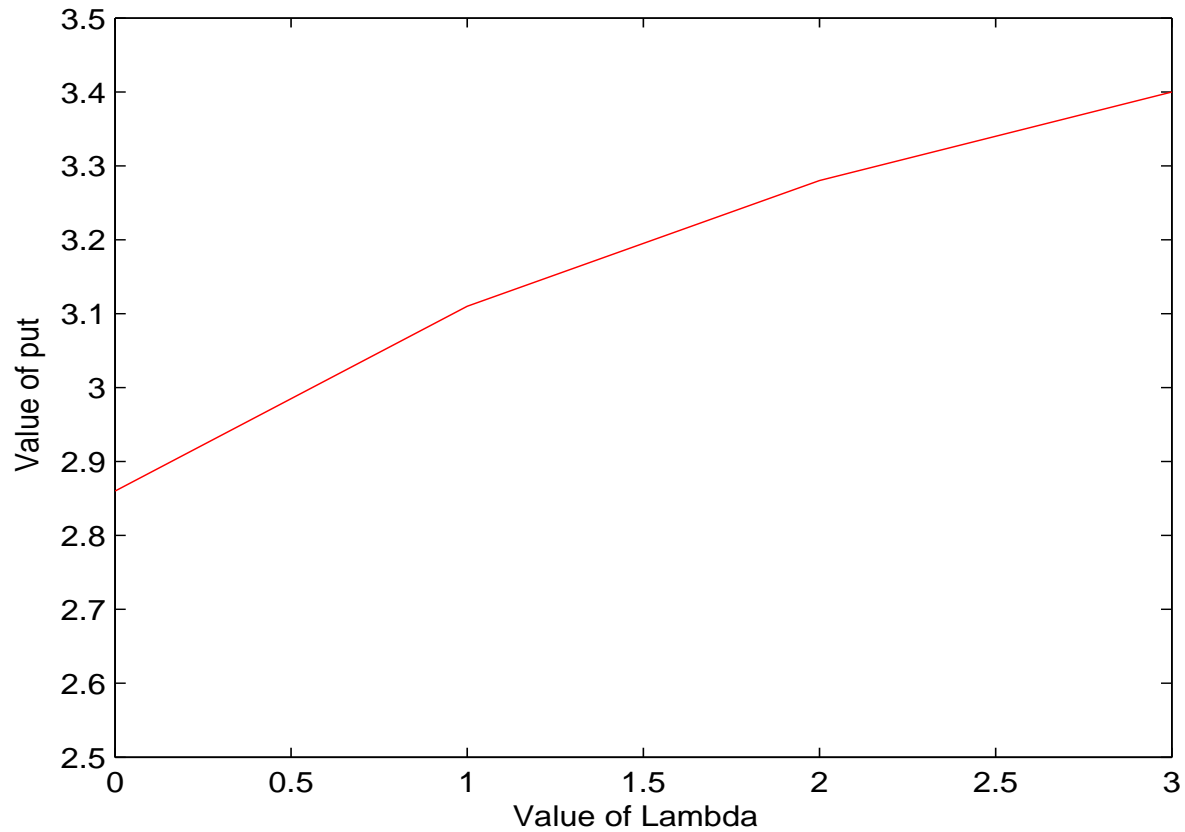


Figure 8: **Put values** for different values of  $\lambda$ . In all cases we assume a 30 basis points contribution fee

## Concluding comments

- the GMWB is a very popular contract.
- It can contain lots of bells and whistles
- Complex product for the consumer to figure
- Value of the put option **very sensitive** to volatility of the underlying portfolio and this is controlled by the contract terms.
- One large insurer currently gives a choice of four funds for its GMWB rider. These funds are
  1. X perspectives
  2. Growth Blend
  3. Value Blend
  4. Growth and IncomeHowever each of these four funds **has 40 percent in a bond fund.**

## Summary

The GMWB is not an easy rider