

**Optimal Asset Allocation and  
Ruin-Minimization Annuitization Strategies:  
The Fixed Consumption Case**

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**Abstract**

In this paper, we derive the optimal investment and annuitization strategy for a retiree whose objective is to minimize the probability of lifetime ruin, namely the probability that a fixed consumption strategy will lead to zero wealth while the individual is still alive. Recent papers in the insurance economics literature have examined utility-maximizing annuitization strategies. Others in the probability, finance, and risk management literature have derived shortfall-minimizing investment and hedging strategies given a limited amount of initial capital. This paper brings the two strands of research together. Our model pre-supposes a retiree who does not currently have sufficient wealth to purchase a life annuity that will yield her exogenously desired fixed consumption level. Therefore, she self-annuitizes, while dynamically managing her investment portfolio and possibly purchasing some annuities to minimize the probability of lifetime ruin. We demonstrate that she will not annuitize any of her wealth until she can fully cover her desired consumption by annuities. We derive a variational inequality that governs the ruin probability and optimal strategies and demonstrate that the problem can be recast as a related optimal stopping problem that yields a more tractable free boundary problem. We numerically approximate the ruin probability and optimal strategies and examine how they change as we vary the mortality assumption and parameters of the financial model. Moreover, we solve the problem implicitly for the special case of exponential future lifetime. As a byproduct, we are able to quantify the reduction in lifetime ruin probability that comes from being able to dynamically manage the investment portfolio.

**JEL Classification:** J26; G11

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## 1. Introduction and Motivation

Global pension reform and the trend towards privatization has focused much academic and practitioner attention on the market for voluntary life annuities, as an alternative to defined benefit pensions. While life annuities themselves are hundreds of years old – see Poterba (1997) for a brief history – it is only recently that they have attracted the attention of noted financial and insurance economists, such as Feldstein and Rangelova (2001), as an alternative to Social Security.

In a well-cited paper from the public economics literature, Yaari (1965) proved that in the absence of annuity bequest motives – and in a deterministic financial economy – consumers will annuitize all of their liquid wealth. Richard (1975) generalized this result to a stochastic environment and a recent paper by Davidoff, Brown, and Diamond (2003) demonstrates the robustness of the Yaari (1965) result. In practice, of course there are market imperfections, and frictions preclude full annuitization results. Similarly, Brugiavini (1993), Kapur and Orszag (1999), Brown (2001), and Milevsky and Young (2003) provide theoretical and empirical guidance on the optimal time to annuitize under various market structures.

The common theme of the above-mentioned papers is the presumption of a rational utility-maximizing economic agent with rigid inter-temporal preferences and pre-specified relative risk aversion. While this von-Neumann-Morgenstern framework is the basis of most of micro-economic foundations, it is notably difficult to apply as a tool for normative advice.

Recently, though, a variety of papers in the risk and portfolio management literature have revitalized the Roy (1950) Safety-First “rule” and applied the concept to probability maximization of achieving certain investment goals. For example, Browne (1995 and 1999a, b, c) derives the optimal dynamic strategy for a portfolio manager who is interested in minimizing the probability of shortfall. Within the insurance arena, Møller (2001) develops risk-minimizing hedging strategies for savings policies with random death benefits.

Indeed, there is something intuitively appealing about minimizing the probability of shortfall that lends itself to asset allocation advice. In fact, in the U.S., the Nobel laureate Bill Sharpe has founded a financial services advisory firm that is largely based on using probabilities to provide investment advice.

Therefore, motivated by the desire to apply the probability minimization concept to the retirement and annuity literature, in this paper we find the optimal annuity-purchasing scheme for an individual who seeks to minimize the probability that she outlives her wealth, also called the probability of lifetime ruin. In other words, we assume the retiree would

like to maintain a pre-specified (exogenous) consumption level, and we provide guidance on the optimal investment strategy, as well as the optimal time to annuitize, in order to minimize the probability that wealth will reach zero while the individual is still alive.

Milevsky and Robinson (2000) introduced the probability of lifetime ruin as a risk-metric for retirees, albeit in a static environment. Similarly, Young (2003) locates the optimal investment policy for an individual who targets a specific consumption rate, who invests in a complete financial market, and who does not buy annuities. By contrast, we allow the individual to buy annuities, as well as to invest in a financial market. The irreversibility of annuity purchases and their illiquidity creates a complex optimization environment, which renders many classical results inoperable. Of course, these same challenges are what make the problem mathematically interesting.

Our agenda for this paper is as follows. In Section 2, we introduce and motivate the concept of self-annuitization and provide some general statements about the probability of lifetime ruin under such a strategy. We, then, present our formal optimization model and prove that if the derivative of the probability of lifetime ruin with respect to annuity income is greater than the adjusted derivative with respect to wealth, the individual will annuitize a lump sum. Thereafter, she will buy annuities at a continuous rate – which we will prove later is zero – in order to keep the derivative of the probability of lifetime ruin with respect to annuity income no greater than the adjusted derivative with respect to wealth. The derivative of the probability of lifetime ruin with respect to wealth is adjusted by multiplying by the price of the annuity purchase factor at any given age. Thus, the annuity-purchasing problem is qualitatively similar to the problem of optimal consumption and investment in the presence of proportional transaction costs. We apply techniques from optimal stochastic control. Friedman and Shen (2002) recently applied similar methods to problems in retirement planning and insurance.

In Section 3, we reduce the dimension of the variational inequality obtained in Section 2. In Section 2, the probability of lifetime ruin is given as a function of the current time, the wealth  $w$  at that time, and the annuity income  $A$  at that time. If  $c$  denotes the desired consumption rate, then it turns out the probability of lifetime ruin is a function of  $z = w/(c - A)$  and time, so we can reduce the dimension of the problem by one. We, then, study properties of the optimal consumption, investment, and annuity-purchasing policies. We show that if the wealth-to-desired additional consumption (or desired consumption minus income) ratio is greater than or equal to the actuarial present value of a continuous annuity that pays \$1 per year, then the individual will purchase a lump sum annuity to guarantee her desired consumption rate so that she will never ruin. Conversely, if the wealth-to-(consumption minus income) ratio is less than the actuarial present value of the

continuous annuity, then the individual will buy no annuity at that time but wait until wealth is great enough, a rather surprising bang-bang result that stands in contrast to the gradual policy that would apply if the objective function were a smooth utility function (Milevsky and Young, 2003).

In Section 4, we linearize the nonlinear partial differential equation for the probability of lifetime ruin via duality techniques. Section 5 formulates the dual problem from Section 4 as an optimal stopping problem. We, then, compare the minimum lifetime ruin probability for static and dynamic strategies for general force of mortality. In Section 6, we examine the annuity-purchasing problem for the specific case of an individual with constant force of mortality, which is synonymous with an exponential distribution for future lifetime. We derive an implicit analytic solution to the problem and provide some numerical examples with a discussion. Section 7 concludes the paper.

## 2. Probability of Lifetime Ruin and Self Annuitization

### 2.1. Deterministic Returns

We start with a future lifetime random variable  $\tau_d$  that is exponentially distributed, for which the probability of survival is given by

$$\Pr[\tau_d > t] = e^{-\lambda t},$$

in which  $\lambda$  is the instantaneous hazard rate (i.e., the force of mortality). The greater the hazard rate  $\lambda$ , the lower is the probability of survival to any given age  $t$ .

Under an exponential mortality assumption the expected (or mean) future lifetime is equal to

$$E[\tau_d] = \int_0^{\infty} e^{-\lambda t} dt = \frac{1}{\lambda}.$$

Thus, for example, if  $\lambda = 0.05$ , then the life expectancy is 20 years, while if  $\lambda = 0.1$ , the life expectancy is 10 years. Note, however, that the median future lifetime – which is distinct from the mean future lifetime – is the value  $m$  of  $t$  at which the probability of survival is exactly equal to 50%. Both median and mean lifetime are valid measures of central tendency for human mortality, and both are used in daily practice (unfortunately with much confusion at times). Thus, inverting the survival probability equation to obtain the 50% mark leads to:

$$0.5 = e^{-\lambda m} \text{ implies } m = \frac{\ln(2)}{\lambda} < \frac{1}{\lambda}.$$

For example, when  $\lambda = 0.05$ , the median future lifetime  $m = \ln(2)/0.05 = 13.862$  years, and when  $\lambda = 0.1$ , we get that  $m = \ln(2)/0.1 = 6.931$  years. Note that  $m$  is always less than life expectancy, as evidenced by the above inequality. The distinction between median and mean lifetime is critical for understanding lifetime ruin probabilities. Recall that 50% of the current population cohort is dead at the median lifetime point, while the other 50% survives. However, the probability of surviving to the mean lifetime of  $1/\lambda$  is a much lower  $e^{-\lambda/\lambda} = e^{-1} = 36.78\%$ . The skewness or asymmetry in the future lifetime random variable can be measured by the gap between the mean and median lifetime, which is  $(1 - \ln(2))/\lambda$ . The lower the hazard rate, the greater the asymmetry.

In this paper, we will be referring to life annuity prices. Formally, the price at age  $x$  of a fixed \$1 per annum payout annuity (for life, with no guarantee period) that is paid continuously is computed via:

$$\int_0^{\infty} e^{-rt} \Pr[\tau_d > t] dt.$$

It is effectively equal to the present value of \$1 per annum discounted by the risk-free rate and the probability of survival. The greater the interest rate, the lower the present value cost of the fixed annuity. In the event of exponential mortality with hazard rate  $\lambda$ , the annuity price equals

$$\int_0^{\infty} e^{-rt} e^{-\lambda t} dt = \frac{1}{r + \lambda}.$$

Thus, for example, if the hazard rate is  $\lambda = 0.05$  (which means that life expectancy is 20 years) and the interest rate in the (annuity) market is  $r = 0.07$ , the price of \$1 for life is  $1/(0.12) = 8.33$  dollars. Stated differently, one dollar of initial premium will yield a fixed annuity payout of  $\lambda + r = 0.12$  dollars per annum for life. However, when the (initial) life expectancy is only 10 years, which implies the hazard rate is  $\lambda = 0.10$ , then under an  $r = 0.07$  interest rate, the cost of \$1 for life is only  $1/(0.17) = 5.88$  dollars. Equivalently, the payout per initial dollar of premium is 0.17 dollars per annum.

In the deterministic case, we use the function  $W(t)$  to denote the wealth at time  $t$  of the retiree assuming she does not annuitize. Instead, she consumes a constant amount  $c$  per annum until she either runs out of money or she dies (which ever comes first). In this subsection, we quantify the dynamics of the wealth process and compute the probability she will run out of money while she is still alive. To simplify our work, we assume only one interest rate  $r$  (i.e., no term structure or expenses) in the economy and that all annuities are priced (fairly) as a function of the hazard rate only (i.e., we ignore loading and expenses).

Formally, under a self-annuitization strategy, the wealth process of the retiree will

obey the ordinary differential equation:

$$dW(t) = (rW(t) - c)dt, \quad W(0) = 1.$$

The individual retires with \$1 of wealth, invests at a rate of  $r$  and consumes at a rate of  $c$ . Intuitively, therefore, wealth increases at the interest rate at which money is invested minus the consumption rate. The solution to this ordinary differential equation is

$$W(t) = \left(1 - \frac{c}{r}\right) e^{rt} + \frac{c}{r}, \quad t \leq t^*,$$

and 0 after time  $t^*$ , in which  $t^*$  is the point at which the process hits zero (i.e., the individual is ruined).

Now, back to the self-annuitization fixed annuity case, assume the consumption rate is set equal to exactly  $c = \lambda + r$ , which is the fixed annuity amount. In this case, the function for wealth is:

$$W(t) = -\frac{\lambda}{r} e^{rt} + \frac{\lambda}{r} + 1, \quad t \leq t^*.$$

More importantly, the rate at which wealth evolves, the derivative of the wealth function, is  $-\lambda e^{rt}$ , which is always negative, and more so the greater the value of  $r$ . In other words, the greater the interest rate, the higher is the rate at which wealth is depleted under a self-annuitization strategy. This is critical for understanding why the interest rate has such a strong impact on the ruin time, even though the life expectancy (or hazard rate) does not change.

Finally, in this self-annuitization case the ruin time  $t^*$ , the point at which the function  $W(t)$  reaches zero, can be simplified to

$$t^* = \frac{1}{r} \ln \left(1 + \frac{r}{\lambda}\right).$$

Note that the derivative of  $t^*$  with respect to  $r$  is negative; that is, as the interest rate decreases, the time of ruin increases.

For example, if  $r = 0.07$  and  $\lambda = 0.05$  (a life expectancy of 20 years), then the ruin time is  $t^* = \ln(1 + 0.07/0.05)/0.07 = 12.506$  years. However, if the interest rate is exactly  $r = 0.05$ , then the ruin time is  $t^* = \ln(1 + 0.05/0.05)/0.05 = 13.862$  years. In words, when the pricing interest rate is reduced – and the replicating consumption strategy is reduced accordingly – the ruin time is later. Note that when  $r = \lambda$ , the value of  $t^* = \ln(2)/\lambda$ , which is exactly the median life span. In other words, when the interest rate is equal to the hazard rate the money runs out at median life expectancy. Finally, in the limit, as  $r$  goes to zero, the ruin time is precisely the life expectancy  $1/\lambda$  because

$$\lim_{r \rightarrow 0^+} \frac{1}{r} \ln \left(1 + \frac{r}{\lambda}\right) = \frac{1}{\lambda}.$$

In other words, the money runs out earlier than life expectancy when the interest rate is positive. Combining the survival equation and the ruin equation, the probability of surviving to the point at which the funds are (exactly) exhausted is

$$\exp(-\lambda t^*) = \exp\left(-\frac{\lambda}{r} \ln\left(1 + \frac{r}{\lambda}\right)\right).$$

First, when  $\lambda = r$ , which means that the hazard rate is exactly equal to the pricing interest rate, the probability of ruin (i.e., being alive when the money runs out) is exactly equal to 50%. In fact, this is the only case it is equal to 50%. Furthermore, when the interest rate is higher than the hazard rate, the probability of lifetime ruin is greater than 50%, and when the interest rate is lower than the hazard rate, the probability of lifetime ruin is lower than 50%. At the extreme, when the interest rate converges to zero, the probability of ruin is 0.3678. Recall that the probability of surviving to the life expectancy point is also equal to 0.3678, and thus we obtain the interesting result that the best odds one can obtain from a self-annuitization strategy – with exponential lifetime and fixed payouts – is precisely  $e^{-1}$ . Later in the paper, we will contrast this deterministic return case with one for which we allow random returns.

## 2.2. Stochastic Returns

In this subsection, we formalize the optimal annuity-purchasing and optimal investment problem for an individual who seeks to minimize the probability that she outlives her wealth. A priori, we allow the individual to buy annuities in lump sums or continuously, whichever is optimal. Our results are similar to those of Dixit and Pindyck (1994, pp 359ff), which are given in the context of real options. They consider the problem of a firm's (irreversible) capacity expansion. For our individual, annuity purchases are also irreversible, and this leads to the similarity in results.

We assume that the individual can invest in a riskless asset whose price at time  $s$ ,  $X_s$ , follows the process  $dX_s = rX_s ds$ ,  $X_t = x > 0$ , for some fixed  $r \geq 0$ , as in the previous subsection. However, unlike the previous subsection, the individual can invest in a risky asset whose price at time  $s$ ,  $S_s$ , follows geometric Brownian motion given by

$$\begin{cases} dS_s = \mu S_s ds + \sigma S_s dB_s, \\ S_t = S > 0, \end{cases}$$

in which  $\mu > r$ ,  $\sigma > 0$ , and  $B_s$  is a standard Brownian motion with respect to a filtration  $\{F_s\}$  of the probability space  $(\Omega, F, \Pr)$ . Let  $W_s$  be the wealth at time  $s$  of the individual (after purchasing annuities at that time), and let  $\pi_s$  be the amount that the decision maker

invests in the risky asset at time  $s$ . We allow annuity purchasing to occur in lump sums, if that is optimal. It follows that the amount invested in the riskless asset is  $W_s - \pi_s$ . Also, the decision maker consumes at a constant rate of  $c$ . Our economy can be either real or nominal. When our model is interpreted in nominal terms, then the consumption rate  $c$  is nominal, and we assume that the individual buys annuities that pay a fixed nominal amount. In practice, of course, this exposes the retiree to inflation risk since  $c$  today will buy much more than  $c$  in twenty years. However, if  $c$  is real, then we assume that the individual (only) has access to annuities that are indexed to inflation and thereby pay a fixed real amount. Also, in this case, the returns on the riskless and risky assets are stated in real returns. We prefer to think of the model in real terms, and our numerical examples are presented in real terms so that inflation risk which would be a problem if  $c$  were stated in nominal terms is not an issue.

As for the actuarial assumptions, let  ${}_t p_x^S$  denote the subjective conditional probability that an individual aged ( $x$ ) believes he or she will survive to age ( $x+t$ ). It is defined via the subjective hazard function,  $\lambda_{x+s}^S$ , by the formula  ${}_t p_x^S = \exp\left(-\int_0^t \lambda_{x+s}^S\right)$ . See Bowers et al. (1997) for further details on this notation. We have a similar formula for the objective conditional probability of survival,  ${}_t p_x^O$  in terms of the objective hazard function,  $\lambda_{x+s}^O$ . The actuarial present value of a life annuity that pays \$1 per year continuously to ( $x$ ) is written  $\bar{a}_x$ . It is defined by  $\bar{a}_x = \int_0^\infty e^{-rt} {}_t p_x dt$ . Note that these formulas all generalize those of the previous subsection.

If we use the subjective hazard rate to calculate the survival probabilities, then we write  $\bar{a}_x^S$ , while if we use the objective (pricing) hazard rate to calculate the survival probabilities, then we write  $\bar{a}_x^O$ . Just to clarify, by objective  $\bar{a}_x^O$ , we mean the actual market prices of the annuity, whereas  $\bar{a}_x^S$  denotes what the market price “would be” were the insurance company to use the individual’s subjective assessment of her mortality. We deliberately refrain from getting into a discussion of insurance anti-selection issues, which is what creates the wedge between subjective and objective hazard rates. In addition, we omit any mention of actuarial loading fees, agent commissions, and other market imperfections that only add to the cost of annuities and can always be absorbed in the subjective hazard rate.

The individual has a non-negative (annuity) income rate at time  $s$  of  $A_s$  after any annuity purchases at that time ( $A_{s-}$  before any annuity purchases then). The exogenous initial income could include Social Security benefits and defined benefit pension benefits, for example. We assume that she can purchase an annuity at the (unloaded) price of  $\bar{a}_{x+s}^O$  per dollar of annuity income at time  $s$ , or equivalently, at age  $x+s$ . Thus, wealth follows

the process

$$\begin{cases} dW_s = [rW_{s-} + (\mu - r)\pi_{s-} + A_{s-} - c] ds + \sigma\pi_{s-}dB_s - \bar{a}_{x+s}^O dA_s, \\ W_{t-} = w \geq 0. \end{cases} \quad (2.1)$$

The negative sign for the subscript on the random processes denotes the left-hand limit of those quantities before any (lump-sum) annuity purchases.

We assume that the decision maker seeks to minimize, over admissible  $\{\pi_s, A_s\}$ , her subjective probability of ruin, namely, the probability that her wealth drops to zero before she dies. Admissible  $\{\pi_s, A_s\}$  are those that are measurable with respect to the information available at time  $s$ , namely  $F_s$ , that restrict the annuity-income process to be non-negative and non-decreasing (i.e., annuity purchases are irreversible), and that result in (2.1) having a unique solution; see Karatzas and Shreve (1998), for example. We also allow the individual to value her probability of lifetime ruin via her subjective hazard rate (or force of mortality), while the annuity is priced by using the objective hazard rate. This is quite intuitive since lifetime ruin is by definition a subjective metric.

Denote the random time of death of our individual by  $\tau_d$ , as in the previous subsection, and the random time of lifetime ruin by  $\tau_0$ ; that is,  $\tau_0$  is the time at which wealth reaches zero. Thus, the probability of lifetime ruin  $\psi$  for the individual at time  $t$ , or age  $x + t$ , defined on  $\bar{D} = \{(w, A, t) : 0 \leq w \leq (c - A)\bar{a}_{x+t}^O, 0 \leq A \leq c, t \geq 0\}$  is given by

$$\psi(w, A, t) = \inf_{\{\pi_s, A_s\}} \Pr[\tau_0 < \tau_d | W_{t-} = w, A_{t-} = A]. \quad (2.2)$$

Note that if  $w \geq (c - A)\bar{a}_{x+t}^O$ , then the individual can purchase an annuity that will guarantee her an income of  $(c - A)$  that added to her income of  $A$  gives her income to match her consumption rate of  $c$ . Thus,  $\psi(w, A, t) = 0$  for  $w \geq (c - A)\bar{a}_{x+t}^O$ . If life annuities were not available, securing lifetime income would necessitate acquiring a perpetuity, which would cost  $1/r$  per \$1 of income, much more than the annuity. This was the problem analyzed by Young (2003).

We continue with a formal discussion of the derivation of the associated HJB equation. Suppose that at the point  $(w, A, t)$ , it is optimal *not* to purchase any annuities. It follows from Itô's lemma that  $\psi$  satisfies the equation at  $(w, A, t)$  given by

$$\lambda_{x+t}^S \psi = \psi_t + (rw + A - c)\psi_w + \min_{\pi} \left[ (\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right]. \quad (2.3)$$

Because the above policy is in general suboptimal, (2.3) holds as an inequality; that is, for all  $(w, A, t)$ ,

$$\lambda_{x+t}^S \psi \leq \psi_t + (rw + A - c)\psi_w + \min_{\pi} \left[ (\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right]. \quad (2.4)$$

Next, assume that at the point  $(w, A, t)$ , it is optimal to buy an annuity instantaneously. In other words, assume that the investor moves instantly from  $(w, A, t)$  to  $(w - \bar{a}_{x+t}^O \Delta A, A + \Delta A, t)$ . Then, the optimality of this decision implies that

$$\psi(w, A, t) = \psi(w - \bar{a}_{x+t}^O \Delta A, A + \Delta A, t), \quad (2.5)$$

which in turns yields

$$\bar{a}_{x+t}^O \psi_w(w, A, t) - \psi_A(w, A, t) = 0. \quad (2.6)$$

Note that the lump-sum purchase is such that the derivative of the probability of lifetime ruin with respect to annuity income equals the adjusted derivative with respect to wealth, in which we adjust by the cost of \$1 of annuity income  $\bar{a}_{x+t}^O$ . This is parallel to many results in economics. Indeed, the derivative of the probability of lifetime ruin with respect to annuity income can be thought of as (the negative of) the marginal utility of the benefit, while the adjusted derivative with respect to wealth can be thought of as (the negative of) the marginal utility of the cost. We say ‘negative’ here because  $\psi$  is decreasing with respect to  $w$  and  $A$ . Thus, the lump-sum purchase forces the marginal utilities of benefit and cost to equal.

However, such a lump-sum purchasing policy is in general suboptimal; therefore, (2.6) holds as an inequality and becomes

$$\bar{a}_{x+t}^O \psi_w(w, A, t) - \psi_A(w, A, t) \leq 0. \quad (2.7)$$

By combining (2.4) and (2.7), we obtain the HJB equation (2.8) below associated with the probability of ruin  $\psi$  given in (2.2). The following result can be proved as in Zariphopoulou (1992), for example.

**Proposition 2.1:** *The probability of lifetime ruin is a constrained viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} & \max \left[ \lambda_{x+t}^S \psi - \psi_t - (rw + A - c)\psi_w - \min_{\pi} \left( (\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right), \bar{a}_{x+t}^O \psi_w - \psi_A \right] \\ & = 0. \end{aligned} \quad (2.8)$$

In the next section, we show that  $(w, A)$  lies on the graph of (2.6) when  $w = (c - A)\bar{a}_{x+t}^O$ . If wealth and annuity income initially lie to the right of the barrier at time  $t$ , i.e.,  $w > (c - A)\bar{a}_{x+t}^O$ , then the individual will immediately spend a lump sum of wealth to guarantee that the probability of lifetime ruin is zero. See Figure 2.1. Otherwise, the annuity income is constant when wealth is low enough, i.e.,  $w < (c - A)\bar{a}_{x+t}^O$ . Once wealth is high enough, i.e.,  $w = (c - A)\bar{a}_{x+t}^O$ , the individual will spend her wealth to guarantee an income rate of  $(c - A) + A = c$  to match her consumption rate of  $c$ .

**Figure 2.1 about here**

Thus, as in Dixit and Pindyck (1994, pp 359ff) or in Zariphopoulou (1992), we have discovered that the optimal annuity-purchasing scheme is a type of barrier control. Other barrier control policies appear in finance and insurance. In finance, Zariphopoulou (1999, 2001) reviews the role of barrier policies in optimal investment in the presence of transaction costs; also see the references within her two articles. See Gerber (1979) for a classic text on risk theory in which he includes a section on optimal dividend payout and shows that it follows a type of barrier control. See Neuberger (2002) for an analysis that is similar to ours.

### 3. Reducing the Dimension of the Minimization Problem

It turns out that the probability of lifetime ruin  $\psi$  is a function of the ratio  $z = w/(c - A)$  and time  $t$ . This observation is also made in Milevsky and Robinson (2000), where the probability of lifetime ruin is shown to depend only on the ratio of current wealth to desired consumption. The easiest way to see this homogeneity is to define  $\tilde{\psi}(w, \tilde{A}, t) = \psi(w, c - \tilde{A}, t)$ . Then,  $\psi(w, A, t) = \tilde{\psi}(w, c - A, t) = \tilde{\psi}(z(c - A), c - A, t) = \tilde{\psi}(z, 1, t)$  with targeted consumption  $c/(c - A)$ , in which the last equality follows from scaling the entire problem by  $(c - A)$ . Thus, define  $V$  by

$$V(z, t) = \tilde{\psi}(z, 1, t), \tag{3.1}$$

so that  $\psi(w, A, t) = V(z, t)$ ,  $\psi_t = V_t$ ,  $\psi_w = \frac{1}{c-A}V_z$ ,  $\psi_{ww} = \left(\frac{1}{c-A}\right)^2 V_{zz}$ , and  $\psi_A = \frac{z}{c-A}V_z$ . Then, the barrier equation in (2.6) becomes

$$zV_z = \bar{a}_{x+t}^O V_z;$$

thus, either  $V_z = 0$  at the barrier or  $z = \bar{a}_{x+t}^O$  there. If one assumes that  $V_z = 0$ , we obtain a contradiction, and we omit the proof of this. Therefore, we have  $z = \bar{a}_{x+t}^O$  at the barrier and  $z < \bar{a}_{x+t}^O$  in the region for which annuity buying is not optimal.

We have just argued that the individual will buy *no* annuities unless  $w \geq (c-A)\bar{a}_{x+t}^O$ , in which case the individual will spend at least  $(c-A)\bar{a}_{x+t}^O$  to buy an annuity guarantee income of  $(c-A)$  from to the annuity. This income plus the income  $A$  covers the consumption rate  $c$ , and the individual will not ruin. Therefore, the individual will not buy any annuity until she can guarantee that she will not ruin, a type of “bang-bang” strategy.

From our preceding discussion we have the following proposition.

**Proposition 3.1:** *For each value of  $t \geq 0$ ,*

- (i) *If  $z \geq \bar{a}_{x+t}^O$ , then the individual immediately buys an annuity to guarantee total income of at least  $c$ ;*
- (ii) *If  $z < \bar{a}_{x+t}^O$ , then the individual buys no annuity; i.e., she is in the region of inaction.*

*It follows that at each time point, the barrier is a ray emanating from the origin and lying in the first quadrant of  $(w, c-A)$  space. Equivalently, in  $(w, A)$  space, the barrier is the ray in the first quadrant with equation  $A = c - w/\bar{a}_{x+t}^O$ .*

Davis and Norman (1990) and Shreve and Soner (1994) find a similar result for the problem of optimal consumption and investment in the presence of proportional transaction costs.

It follows that the HJB equation for  $\psi$  from Proposition 2.1 becomes the following equation for  $V$ :

$$\max \left[ \lambda_{x+t}^S V - V_t - (rz - 1)V_z - \min_{\hat{\pi}} \left( (\mu - r)\hat{\pi}V_z + \frac{1}{2}\sigma^2\hat{\pi}^2V_{zz} \right), z - \bar{a}_{x+t}^O \right] = 0, \quad (3.2)$$

in which  $\hat{\pi} = \frac{\pi}{c-A}$ . Davis and Norman (1990) and Shreve and Soner (1994) use a similar transformation in the problem of consumption and investment in the presence of transaction costs. Also, Duffie et al. (1997) and Koo (1998) use a similar transformation to study optimal consumption and investment with stochastic income.

We are now ready to give a complete formulation of the probability of lifetime ruin  $\psi$ .

**Proposition 3.2:** *The probability of lifetime ruin  $\psi$  in (2.2) is given by*

$$\psi(w, A, t) = V(z, t) \text{ if } z := w/(c-A) < \bar{a}_{x+t}^O; \text{ otherwise, } \psi(w, A, t) = 0,$$

*in which  $V$  solves*

$$\lambda_{x+t}^S V = V_t + (rz - 1)V_z + \min_{\hat{\pi}} \left( (\mu - r)\hat{\pi}V_z + \frac{1}{2}\sigma^2\hat{\pi}^2V_{zz} \right), \quad (3.3)$$

*for  $z < \bar{a}_{x+t}^O$ , with boundary conditions  $V(0, t) = 1$  and  $V(\bar{a}_{x+t}^O, t) = 0$  and with transversality condition  $\lim_{s \rightarrow \infty} s_{-t} p_{x+t}^S E[V(Z_s^*, s) | Z_t = z] = 0$ , in which  $Z_s^*$  is the optimally controlled  $Z_s$ .*

#### 4. Linearizing the Equation for $V$ via Duality Arguments

In this section, we linearize the nonlinear partial differential equation for  $V$  in equation (3.3). To do this, we first eliminate the  $\lambda_{x+t}^S V$  term from (3.3) by defining

$$f(z, t) = {}_t p_x^S V(z, t).$$

It follows that (3.3) becomes

$$f_t + (rz - 1)f_z + \min_{\hat{\pi}} \left[ (\mu - r)\hat{\pi}f_z + \frac{1}{2}\sigma^2\hat{\pi}^2 f_{zz} \right] = 0, \quad (4.1)$$

with boundary conditions  $f(0, t) = {}_t p_x^S$  and  $f(\bar{a}_{x+t}^O, t) = 0$  and with transversality condition  $\lim_{s \rightarrow \infty} E[f(Z_s^*, s) | Z_t = z] = 0$ . This condition can be rewritten as  $\lim_{t \rightarrow \infty} f(z, t) = 0$  with probability 1 because  $0 \leq f \leq 1$ .

Next, consider the concave dual of  $f$  defined by

$$\tilde{f}(y, t) = \min_{z > 0} [f(z, t) + zy]. \quad (4.2)$$

The critical value  $z^*$  solves the equation  $f_z(z, t) + y = 0$ ; thus,  $z^* = I(-y, t)$ , in which  $I$  is the inverse of  $f_z$  with respect to  $z$ . It follows that

$$\tilde{f}(y, t) = f[I(-y, t), t] + yI(-y, t). \quad (4.3)$$

Note that

$$\begin{aligned} \tilde{f}_y(y, t) &= -f_z[I(-y, t)]I_y(-y, t) + I(-y, t) - yI_y(-y, t) \\ &= yI_y(-y, t) + I(-y, t) - yI_y(-y, t) \\ &= I(-y, t). \end{aligned} \quad (4.4)$$

We can retrieve the function  $f$  from  $\tilde{f}$  by the relationship

$$f(z, t) = \max_{y > 0} [\tilde{f}(y, t) - zy]. \quad (4.5)$$

Indeed, the critical value  $y^*$  solves the equation  $\tilde{f}_y(y, t) - z = 0$ ; thus,  $y^* = -f_z(z, t)$ , and

$$\begin{aligned} \tilde{f}(y^*, t) - zy^* &= f[I(-y^*, t), t] + y^*I(-y^*, t) - zy^* \\ &= f[I(f_z(z, t), t), t] - f_z(z, t)I(f_z(z, t), t) + zf_z(z, t) \\ &= f(z, t) - zf_z(z, t) + zf_z(z, t) \\ &= f(z, t), \end{aligned}$$

in which we use equation (4.3) for the first equality.

Next, note that

$$\tilde{f}_{yy}(y, t) = -I_y(-y, t) = -1/f_{zz}[I(-y, t), t], \quad (4.6)$$

and

$$\begin{aligned} \tilde{f}_t(y, t) &= f_z[I(-y, t), t]I_t(-y, t) + f_t[I(-y, t), t] + yI_t(-y, t) \\ &= -yI_t(-y, t) + V_t[I(-y, t), t] + yI_t(-y, t) \\ &= f_t[I(-y, t), t]. \end{aligned} \quad (4.7)$$

In the partial differential equation for  $f$ , let  $z = I(-y, t)$  to obtain

$$f_t[I(-y, t), t] + (rI(-y, t) - 1)f_z[I(-y, t), t] - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \frac{(f_z[I(-y, t), t])^2}{f_{zz}[I(-y, t), t]} = 0.$$

Rewrite this equation in terms of  $\tilde{f}$  to get

$$\tilde{f}_t(y, t) + (rI(-y, t) - 1)(-y) - m \frac{(-y)^2}{-1/\tilde{f}_{yy}(y, t)} = 0,$$

in which  $m = \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2$ , or equivalently,

$$\tilde{f}_t(y, t) - ry\tilde{f}_y(y, t) + my^2\tilde{f}_{yy}(y, t) + y = 0, \quad (4.8)$$

with boundary conditions given implicitly by  $f(0, t) = {}_t p_x^S$  and  $f(\bar{a}_{x+t}^O, t) = 0$ . Note that (4.8) is a *linear* partial differential equation.

Now, consider the boundary conditions  $f(0, t) = {}_t p_x^S$  and  $f(\bar{a}_{x+t}^O, t) = 0$ . Because  $f_z < 0$  is strictly increasing with respect to  $z$ , we have  $y_0(t) > y_b(t) \geq 0$  for all  $t \geq 0$ , in which  $y_0(t)$  and  $y_b(t)$  are defined by

$$y_0(t) = -f_z(0, t), \quad (4.9)$$

and

$$y_b(t) = -f_z(\bar{a}_{x+t}^O, t). \quad (4.10)$$

Thus, the boundary conditions become

$$\tilde{f}(y_0(t), t) = {}_t p_x^S, \text{ for } \tilde{f}_y(y_0(t), t) = 0, \quad (4.11)$$

and

$$\tilde{f}(y_b(t), t) = \bar{a}_{x+t}^O \text{ for } \tilde{f}_y(y_b(t), t) = \bar{a}_{x+t}^O. \quad (4.12)$$

The transversality condition  $\lim_{t \rightarrow \infty} f(z, t) = 0$  with probability 1 becomes  $\lim_{t \rightarrow \infty} \tilde{f}(y, t) = 0$  with probability 1. Note that the first equations in (4.11) and (4.12) are reminiscent of value matching conditions, while the second equations are reminiscent of smooth pasting conditions. We exploit this observation in the next section, where we express  $\tilde{f}$  as the value function for an optimal stopping problem. Thus, we are able to solve for  $\tilde{f}$  via a numerical method such as projected SOR (Howison, Wilmott, and Dewynne, ??).

## 5. Optimal Stopping Formulation

### 5.1. Re-interpreting the Value Function

Equations (4.11) and (4.12) motivate us to define a penalty function  $u$  by

$$u(y, t) = \min({}_t p_x^S, \bar{a}_{x+t}^O y). \quad (5.1)$$

We consider this function because it is maximal among those functions that are concave in  $y$  and satisfy the boundary conditions in (4.11) and (4.12). Recall that  $\tilde{f}$  is concave and increasing in  $y$ . Thus,  $\tilde{f}(y, t) \leq u(y, t)$  for all  $(y, t)$  such that  $y_b(t) \leq y \leq y_0(t)$ .

Define a stochastic process  $Y_s$  by

$$\begin{cases} dY_s = -rY_s + \frac{\mu - r}{\sigma} Y_s d\tilde{B}_s \\ Y_t = y > 0. \end{cases} \quad (5.2)$$

Finally, consider the optimal stopping problem given by

$$\hat{f}(y, t) = \inf_{\tau} E \left[ \int_t^{\tau} Y_s ds + u(Y_{\tau}, \tau) | Y_t = y \right]. \quad (5.3)$$

One can think of this problem as awarding a “player” the running penalty  $Y_s$  between time  $t$  and the time of stopping  $\tau$ . At the time of stopping, the player receives the penalty  $u(Y_{\tau}, \tau)$ . Thus, the player has to decide when it is better to continue receiving the running penalty  $Y_s$  or to stop and take the final penalty  $u(Y_{\tau}, \tau)$ .

A candidate solution for  $\hat{f}$  is the value function  $\tilde{f}$  from Section 4. Indeed, Øksendal (2000, Section 10.4) studies such optimal stopping problems and proves a verification theorem that we can apply as follows: If we can show that

$$u_t(y, t) - r y u_y(y, t) + m y^2 u_{yy}(y, t) + y \geq 0, \quad (5.4)$$

for  $y > y_0(t)$  and for  $y < y_b(t)$ , and if  $\tilde{f}$  is sufficiently regular (smooth, etc.), then  $\hat{f} = \tilde{f}$ . Thus, to numerically solve for  $\hat{f}$ , we can use algorithms developed for optimal stopping problems and solve for  $\hat{f}$ , the value function of the optimal stopping problem.

It remains for us to verify that inequality (5.4) holds. Indeed, for  $y < y_b(t)$ , we have that  $u(y, t) = \bar{a}_{x+t}^O y$ , so that

$$\begin{aligned} u_t(y, t) - ryu_y(y, t) + my^2u_{yy}(y, t) + y \\ &= [-1 + (r + \lambda_{x+t}^O)\bar{a}_{x+t}^O]y - ry\bar{a}_{x+t}^O + y \\ &= \lambda_{x+t}^O\bar{a}_{x+t}^O y \geq 0, \end{aligned}$$

so (5.4) holds here. For  $y > y_0(t)$ , we have that  $u(y, t) = {}_t p_x^S$ , so that

$$\begin{aligned} u_t(y, t) - ryu_y(y, t) + my^2u_{yy}(y, t) + y \\ &= -\lambda_{x+t}^S {}_t p_x^S + y, \end{aligned}$$

and this expression is nonnegative for all  $y > y_0(t)$  if and only if

$$y_0(t) \geq \lambda_{x+t}^S {}_t p_x^S,$$

and this is something we will check in the numerical example in the next subsection.

## 5.2. Numerical Example

Suppose that we have the following values of the parameters:

- $\lambda_x^O = \exp((x - m)/b)/b$ , in which  $m$  is a modal value and  $b$  is a scale parameter. Note that the force of mortality increases exponentially with age. Use  $m = 90$  and  $b = 9$ , values similar to those used by Milevsky and Young (2003). Also, let  $\lambda_x^S = \eta\lambda_x^O$ , in which  $\eta > 0$  is a parameter that represents how the individual views their health relative to the average. To begin, we let  $\eta = 1$ .

- $r = 0.02$ ; the riskless rate of return is 2% over inflation.
- $\mu = 0.06$ ; the risky rate of return is 6% over inflation.
- $\sigma = 0.20$ ; the risky assets volatility is 20%.
- $c = 1$ ; the individual consumes one unit of wealth per year.
- $A = 0$ ; without loss of generality, we assume that annuity income is zero. In the experiments that follow, we will examine the impact on the ruin probability and optimal investment strategy of varying individual parameters from the values given above.

Figure 1 shows the ruin probability  $V(z, 0)$  and optimal investment in the risky asset  $\pi^*(z, 0)$  for the base scenario described above as well as for ages 30 and 70. We note that, consistent with the model formulation,  $V(0, 0) = 1$  and  $V(\bar{a}_x, 0) = 0$ . Moreover, we see that the ruin probability and optimal investment in the risky asset decrease as age increases; this result is consistent with our financial intuition.

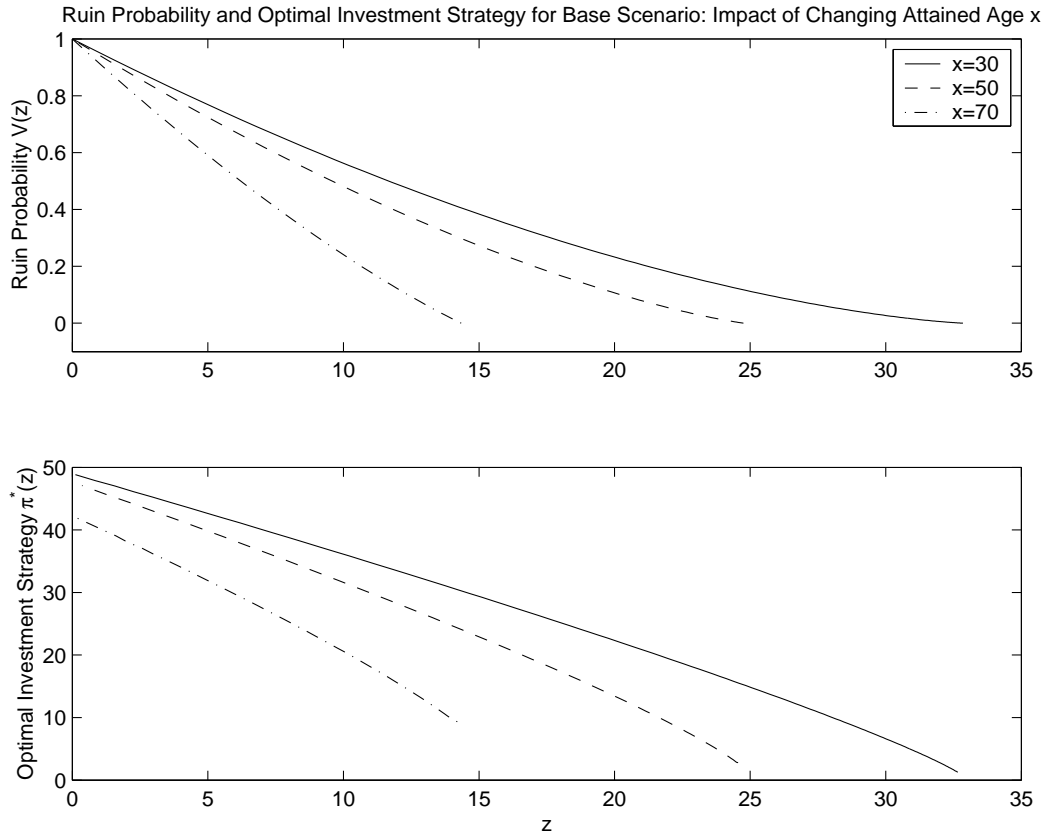


Figure 1: Ruin probabilities and optimal investment strategies as we vary the attained age  $x$

In Figure 2, we examine the impact of changing the volatility  $\sigma$  of the stock return. We observe that for fixed  $z$ , the ruin probability increases and the optimal investment in the risky asset decreases with  $\sigma$ . Again, this is consistent with our financial intuition.

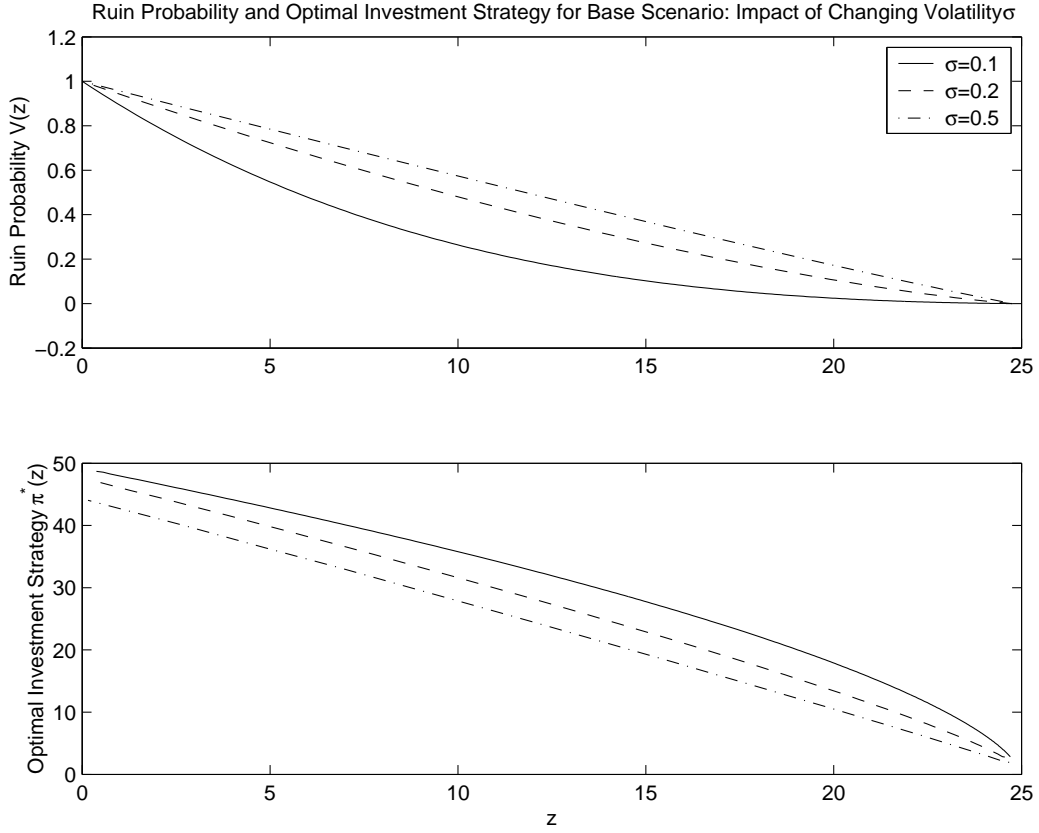


Figure 2: Ruin probabilities and optimal investment strategies as we vary the volatility  $\sigma$

In Figure 3, we vary the subjective mortality parameter  $\eta$ . Since  $\lambda_x^S = \eta \lambda_x^O$ , when  $\eta < 1$ , the individual is healthier than the objective (pricing) mortality suggests. Figure 3 shows that healthier individuals should invest more in the risky asset. This is consistent with the result in Figure 1; if one has a longer horizon, it is optimal to invest more in the risky asset. We remark though that changing  $\eta$  has minimal impact on the ruin probability  $V$ ; in effect, the change in the investment strategy neutralizes the impact of the change in  $\eta$ . We observed similar phenomena in Moore and Young (2003).

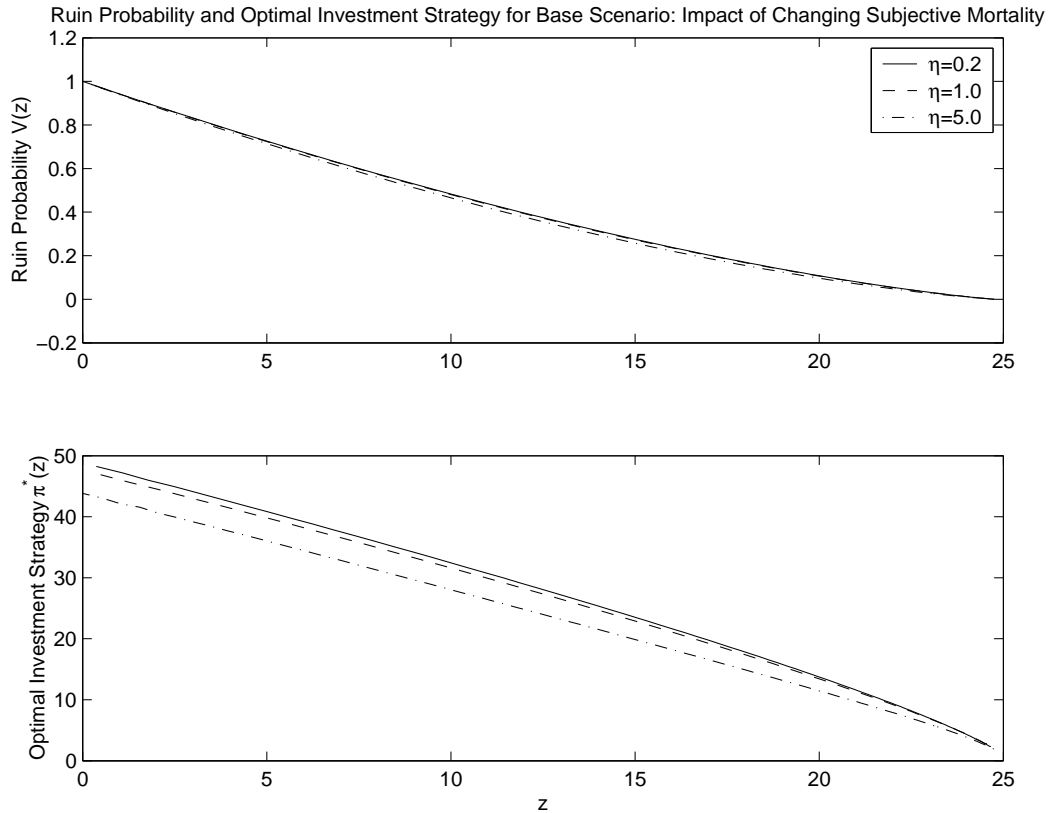


Figure 3: Ruin probabilities and optimal investment strategies as we vary the subjective mortality parameter  $\eta$

In Moore and Young (2004), the authors study minimal ruin probability and optimal strategies in a market in which life annuities are not available. In Figure 4, we compare the ruin probability and optimal investment strategy with those quantities for an investor to whom life annuities are not available. We see that the presence of annuities decreases the probability of lifetime ruin. Indeed, when  $z = \bar{a}_x \approx 24.75$ ,  $V(z, 0) = 0$ ; in a market with no annuities, an investor requires greater wealth to ensure that  $V(z, 0) = 0$ .

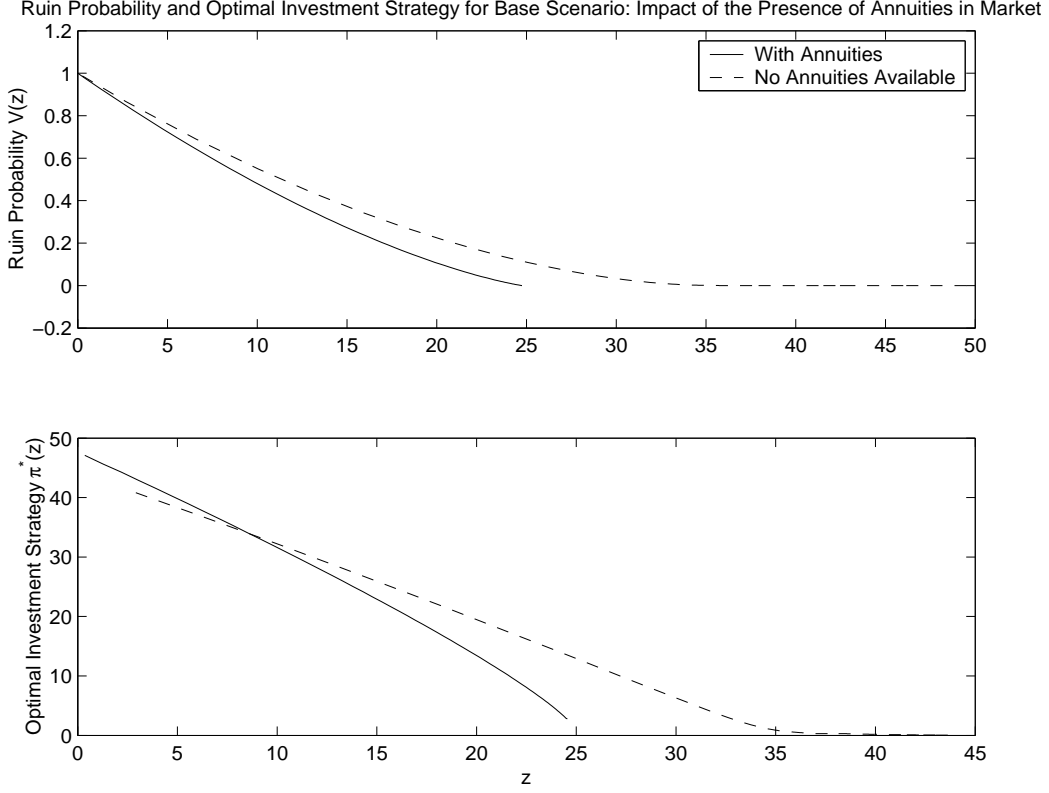


Figure 4: The impact on the ruin probability and optimal investment strategy of the inclusion of annuities in the market

## 6. Constant Force of Mortality

### 6.1. Solution of the Boundary-Value Problem

If we assume that the forces of mortality are constant, that is,  $\lambda_{x+t}^S \equiv \lambda^S$  and  $\lambda_{x+t}^O \equiv \lambda^O$  for all  $t \geq 0$ , then we can obtain an “implicit” analytical solution of the probability of lifetime ruin by calculating the dual of  $V$  in equation (3.3) directly, that is, before removing the  $\lambda_{x+t}^S V$  term. The reason for doing this is that in this case,  $\psi$  and  $V$  are independent of time, so (3.3) becomes the ordinary differential equation:

$$\lambda_{x+t}^S V = (rz - 1)V' + \min_{\hat{\pi}} \left( (\mu - r)\hat{\pi}V' + \frac{1}{2}\sigma^2\hat{\pi}^2V'' \right), \quad (6.1)$$

with boundary conditions  $V(0) = 1$  and  $V(1/(r + \lambda^O)) = 0$ .

If we define the dual of  $V$  by  $\tilde{V}(n) = \min_{z>0}[V(z) + zn]$ , as in Section 4, then we obtain the following boundary-value problem for  $\tilde{V}$ :

$$-\lambda^S \tilde{V}(n) - (r - \lambda^S)y\tilde{V}'(n) + my^2\tilde{V}''(n) + y = 0, \quad (6.2)$$

with boundary conditions

$$\tilde{V}(n_0) = 1, \text{ for } \tilde{V}'(n_0) = 0, \quad (6.3)$$

and

$$\tilde{V}(n_b) = \frac{n_b}{r + \lambda^O}, \text{ for } \tilde{V}'(n_b) = \frac{1}{r + \lambda^O}. \quad (6.4)$$

The general solution of (6.2) is

$$\tilde{V}(n) = D_1 n^{B_1} + D_2 n^{B_2} + \frac{n}{r}, \quad (6.5)$$

with  $D_1$  and  $D_2$  constants to be determined by the boundary conditions, and with  $B_1$  and  $B_2$  given by

$$B_1 = \frac{1}{2m} \left[ (r - \lambda^S + m) + \sqrt{(r - \lambda^S + m)^2 + 4m\lambda^S} \right] > 1, \quad (6.6)$$

and

$$B_2 = \frac{1}{2m} \left[ (r - \lambda^S + m) - \sqrt{(r - \lambda^S + m)^2 + 4m\lambda^S} \right] < 0. \quad (6.7)$$

The boundary conditions at  $n_b$  give us

$$D_1 n_b^{B_1} + D_2 n_b^{B_2} + \frac{n_b}{r} = \frac{n_b}{r + \lambda^O}, \quad (6.8)$$

and

$$D_1 B_1 n_b^{B_1} + D_2 B_2 n_b^{B_2} + \frac{n_b}{r} = \frac{n_b}{r + \lambda^O}. \quad (6.9)$$

Solve equations (6.8) and (6.9) to get  $D_1$  and  $D_2$  in terms of  $n_b$ :

$$D_1 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{1 - B_2}{B_1 - B_2} n_b^{1-B_1} < 0, \quad (6.10)$$

and

$$D_2 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1 - 1}{B_1 - B_2} n_b^{1-B_2} < 0. \quad (6.11)$$

Next, substitute for  $D_1$  and  $D_2$  in the second equation in (5.3), namely  $D_1 B_1 n_0^{B_1-1} + D_2 B_2 n_0^{B_2-1} + \frac{1}{r} = 0$ , to get

$$\frac{\lambda^O}{r + \lambda^O} \frac{B_1(1 - B_2)}{B_1 - B_2} \left( \frac{n_0}{n_b} \right)^{B_1-1} + \frac{\lambda^O}{r + \lambda^O} \frac{B_2(B_1 - 1)}{B_1 - B_2} \left( \frac{n_0}{n_b} \right)^{B_2-1} = 1. \quad (6.12)$$

(6.12) gives us an equation for the ratio  $n_0/n_b > 1$ . To check that (6.12) has a unique solution greater than 1, note that the left-hand side (1) equals  $\lambda^O/(r + \lambda^O) < 1$  when we

set  $n_0/n_b = 1$ , (2) goes to infinity as  $n_0/n_b$  goes to infinity, and (3) is strictly increasing with respect to  $n_0/n_b$ .

Next, substitute for  $D_1$  and  $D_2$  in the first equation in (6.3), namely  $D_1 n_0^{B_1-1} + D_2 n_0^{B_2-1} + \frac{1}{r} = \frac{1}{n_0}$  to get

$$-\frac{\lambda^O}{r(r+\lambda^O)} \frac{1-B_2}{B_1-B_2} \left(\frac{n_0}{n_b}\right)^{B_1-1} - \frac{\lambda^O}{r(r+\lambda^O)} \frac{B_1-1}{B_1-B_2} \left(\frac{n_0}{n_b}\right)^{B_2-1} + \frac{1}{r} = \frac{1}{n_0}. \quad (6.13)$$

Substitute for  $n_0/n_b$  in equation (6.13), and solve for  $n_0$ . Finally, we can get  $n_b$  from

$$n_b = \frac{n_0}{n_0/n_b}, \quad (6.14)$$

and  $D_1$  and  $D_2$  from equations (6.10) and (6.11), respectively.

Once we have the solution for  $\tilde{V}$ , we can recover  $V$  by

$$\begin{aligned} V(z) &= \max_{n>0} [\tilde{V}(n) - zn] \\ &= \max_{n>0} \left[ D_1 n^{B_1} + D_2 n^{B_2} + \frac{n}{r} - zn \right], \end{aligned} \quad (6.15)$$

in which the critical value  $n^*$  solves

$$D_1 B_1 n^{B_1-1} + D_2 B_2 n^{B_2-1} + \frac{1}{r} = z. \quad (6.16)$$

Thus, for a given value of  $z = w/(c-A)$ , solve (6.16) for  $n$  and plug that value of  $n$  into (6.15) to get  $\psi(w, A) = V(z)$ .

Also of interest is the investment in the risky asset, especially as wealth approaches  $(c-A)/(r+\lambda^O)$ .

$$\pi^*(w, A) = (c-A)\hat{\pi}^*(z) = -(c-A) \frac{\mu-r}{\sigma} \frac{V'(z)}{V''(z)} = -(c-A) \frac{\mu-r}{\sigma} n \tilde{V}''(n).$$

Now,  $n \tilde{V}''(n) = D_1 B_1 (B_1-1) n^{B_1-1} + D_2 B_2 (B_2-1) n^{B_2-1}$ , so after substituting for  $D_1$  and  $D_2$  from equations (6.10) and (6.11), respectively, the optimal investment in the risky asset (in terms of  $n$ ) becomes  $(c-A)\hat{\pi}^*(z)|_{z=I(-n)} =$

$$(c-A) \frac{\mu-r}{\sigma} \frac{\lambda^O}{r(r+\lambda^O)} \frac{(B_1-1)(1-B_2)}{B_1-B_2} \left[ B_1 \left(\frac{n}{n_b}\right)^{B_1-1} - B_2 \left(\frac{n}{n_b}\right)^{B_2-1} \right]. \quad (6.17)$$

In particular, as  $n$  approaches  $n_b$ , the point at which the individual annuitizes all her wealth, the amount invested in the risky asset approaches

$$(c-A) \frac{2r}{\mu-r} \left( \frac{1}{r} - \frac{1}{r+\lambda^O} \right), \quad (6.18)$$

independent of  $\sigma$  and  $\lambda^S$ . Note that the expression in (6.18) is a multiple of the difference between the cost of the perpetuity and the cost of the annuity.

In addition to the amount invested in the risky asset, it is useful to know how that amount changes as one's wealth changes. Note that the derivative of  $\pi^*(w, A)$  with respect to  $w$  has the same sign as the derivative of  $n\tilde{V}''(n)$  with respect to  $n$ . Thus, the amount of wealth invested in the risky asset decreases with respect to wealth if and only if

$$n\tilde{V}'''(n) + \tilde{V}''(n) < 0 \text{ for all } n \in (n_b, n_0). \quad (6.19)$$

After some elementary algebra, we determine that (6.19) holds if and only if  $\lambda^S < r$ , while if  $\lambda^S$  is sufficiently larger than  $r$ , then the amount of wealth invested in the risky asset increases with wealth, as we will see in the example in the next section.

## 6.2. Numerical Example

In this section, we present a numerical example to demonstrate the results of Section 6.1. We will calculate the probability of lifetime ruin  $\psi(w, A)$  in the presence of annuities with the corresponding probability  $\psi_0(w, A)$  when the individual cannot buy annuities but has a preexisting income rate of  $A$ , the problem studied in Young (2003). From that work, we know that the probability of lifetime ruin  $\psi_0(w, A)$  is given by

$$\psi_0(w, A) = (1 - rz)^p, \text{ for } 0 \leq z < \frac{1}{r}, \quad (6.20)$$

in which  $z = w/(c - A)$  and

$$p = \frac{1}{2r} \left[ (r + \lambda^S + m) + \sqrt{(r + \lambda^S + m)^2 - 4r\lambda^S} \right] > 1. \quad (6.21)$$

**Example 6.1, Constant Real Dollar Consumed:** Suppose we have the following values of the parameters:

- $\lambda^S = \lambda^O = 0.04$ ; the force of mortality is constant such that the expected future lifetime is 25 years.
- $r = 0.02$ ; the riskless rate of return is 2% over inflation.
- $\mu = 0.06$ ; the risky rate of return is 6% over inflation.
- $\sigma = 0.20$ ; the risky assets volatility is 20%.
- $c = 1$ ; the individual consumes one unit of wealth per year.
- $A = 0$ ; without loss of generality, we assume that annuity income is zero.

The cost of the annuity is  $1/(r + \lambda^O) = 1/0.06 = \$16.666$ , while the cost of the perpetuity is flat at  $1/r = 1/0.02 = \$50$ . Without any loss of generality, we can focus on

the case in which  $A = 0$ . In this example,  $D_1 = -103.4$ ,  $D_2 = -0.002642$ ,  $n_0 = 0.081$ , and  $n_b = 0.044$ . In Table 1, we give the probabilities of ruin  $\psi$  and  $\psi_0$  and the corresponding optimal investments in the risky asset as a proportion of  $(c - A) = 1$ , i.e.,  $\hat{\pi}^*$  and  $\hat{\pi}_0^*$ .

Table 1. Probability of Lifetime Ruin and Optimal Investment in Risky Asset

$z$	$\psi(w, A) = V(z)$	$\hat{\pi}^*(z)$	$\psi_0(w, A) = V_0(z)$	$\hat{\pi}_0^*(z)$
0.0	1.000	25.283	1.000	20.711
0.5	0.960	25.300	0.966	20.504
1.0	0.921	25.327	0.933	20.296
2.0	0.844	25.415	0.870	19.882
5.0	0.633	25.977	0.698	18.640
7.5	0.474	26.829	0.574	17.604
10.0	0.330	28.066	0.467	16.569
12.0	0.223	29.345	0.392	15.740
14.0	0.123	30.885	0.326	14.912
16.0	0.030	32.680	0.268	14.083
16.5	0.0074	33.168	0.255	13.876
16.6	0.00296	33.267	0.252	13.835
16.66	0.000296	33.327	0.251	13.810
16.666	0.0000296	33.333	0.251	13.807
20.0	0.000	n.a.	0.175	12.426

There are a variety of interesting lessons that can be gleaned from the numbers in Table 1. First, for very low values of  $z$ , the ratio of current wealth to the desired additional consumption, the probability of lifetime ruin is (obviously) close to 100%, but it is quite insensitive to whether or not annuities are available. Intuitively, the reason is that the costs of the annuity and the perpetuity, are both relatively far from current wealth and are therefore probabilistically inaccessible. However, as the value of  $z$  increases, the probability of lifetime ruin starts to decline, and the rate of probability improvement is much higher when the life annuity is available. In fact, as we get very close to the cost of the annuity, \$16.666, the probability of lifetime ruin approaches zero – since as soon as that level is breached the entire wealth will be annuitized – while the perpetuity cost is still a distance away at \$50.

As predicted by equation (6.18), as we get (epsilon) close to the annuity cost – i.e. when  $z = 16.5, 16.6$  etc. – we see the equity allocation move towards  $33.333 = 50.00 - 16.666$ , the difference between the cost of the perpetuity and the cost of an annuity. Another use of the results in Table 1 is to invert the  $\psi$  function and solve for the current level of wealth-to-consumption needed to maintain a lifetime ruin probability under some pre-specified level. Thus, for example, if the retiree is interested in having at least a 95% chance of lifetime consumption survival – which implies at most a 5% probability of lifetime ruin – then she must have wealth of at least  $z = 15.55$  times her desired consumption.

Note from Table 1, that  $\psi(w, A) < \psi_0(w, A)$ , as expected, because the probability of lifetime ruin should decrease as the individual's investment opportunities expand to include annuities. On the other hand, the optimal investment in the risky asset increases with respect to  $z$  in the presence of annuities for this particular example (that is,  $\lambda^S = 0.04$  is sufficiently larger than  $r = 0.02$ ), but it decreases when the individual cannot buy annuities. Young (2003) showed the latter, but the behavior of  $\hat{\pi}^*$  might be the opposite when the individual can buy annuities, even though she does not buy any until  $z \geq 1/(r + \lambda^O) = 16.667$ .

The following table illustrates the impact of life expectancy (or current age) on the ruin probability and the optimal allocation to equity. We present three different cases. The first is for a life expectancy of 15 years ( $\lambda = 0.066$ ), the second is a life expectancy of 20 years ( $\lambda = 0.05$ ), and the final is the above-calculated case ( $\lambda = 0.04$ ).

Table 2. The Impact of Life Expectancy on Lifetime Ruin and Optimal Investment

	1/ $\lambda = 15$		1/ $\lambda = 20$		1/ $\lambda = 25$	
Wealth to	Ruin	Equity	Ruin	Equity	Ruin	Equity
Consump'n	Probability	Investment	Probability	Investment	Probability	Investment
Ratio ( $z$ )	$\psi(w, A)$	$\hat{\pi}^*(z)$	$\psi(w, A)$	$\hat{\pi}^*(z)$	$\psi(w, A)$	$\hat{\pi}^*(z)$
10.0	0.111	34.980	0.248	29.978	0.330	28.066
12.0	0.000	n.a.	0.128	32.476	0.223	29.345
14.0	0.000	n.a.	0.016	35.289	0.123	30.885
16.0	0.000	n.a.	0.000	n.a.	0.030	32.680
18.0	0.000	n.a.	0.000	n.a.	0.000	n.a.

Notice from Table 2 that as the individual's life expectancy increases, the amount invested in the risky asset decreases and the probability of lifetime ruin increases, for a given level of  $z$ .

Note: we still need to contrast these results with those from Section 2.1 and thus quantify the effect of dynamic portfolio management. However, I will not focus on these results during my 4/28/04 talk.

## 7. Summary

In this paper we have derived the optimal investment and annuitization strategy for a retiree whose objective is minimize the *probability of lifetime ruin*, the probability that a fixed consumption strategy will lead to zero wealth while the individual is still alive. We obtain a variety of interesting results. First, given the zero-one objective function, we

find that the ruin-minimizing annuitization strategies are of the bang-bang, as opposed to gradual, type. Likewise, when the investment opportunity set is expanded to allow for annuities, one observes a greater allocation to risky assets while waiting to annuitize if the subjective force of mortality is large enough. If the subjective force of mortality is less than the riskless rate of return, then the opposite is true.

Finally, although this paper has focused exclusively on *fixed* payout annuities, our techniques are applicable to *variable* payout products as well. We save that analysis for future work.

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