Asset Location and Allocation with Multiple Risky Assets

Ashraf Al Zaman*

Krannert Graduate School of Management, Purdue University, IN

zamanaa@mgmt.purdue.edu

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Abstract

In this paper, we report the findings of our analysis of asset location and allocation problem with multiple risky assets, where an investor has access to a taxable brokerage account (TBA) and a tax deferred retirement account (TDRA). In the analysis, we capture the interaction of portfolio diversification and the tax timing option. We find that the correlation structure of the risky assets may have substantial impact on the location decision. We also find that asset location and allocation decisions are sensitive to parameter values such as basis-price ratio,

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retirement wealth ratio, Sharpe ratios of the risky assets, and institutional restrictions such as borrowing constraint. Taxable bonds may optimally be located in the TDRA, and equities in the TBA for some reasonable parameter values. But we also observe interesting mix of bonds and equities in both of the accounts with and without borrowing constraints, which is consistent with existing empirical findings. Consumption is mildly affected by investor's retirement wealth ratio, and is substantially affected by the bequest motive during the retirement age. But location and allocation decisions are relatively robust to bequest motives. In a limited scope, we also report the impact of retirement contribution limit on location and allocation decisions, and the interaction between retirement account size and borrowing constraint.

1 Introduction

One of the most important decisions any working adult has to make pertains to the composition of his or her investment portfolio. In making investment decisions, an average working adult is faced with two major questions. The first question relates to asset *allocation*—how should she allocate her investment resources among various risky and risk free assets. The second question relates to asset *location*—given the availability of both taxable and tax-deferred savings accounts, how should she locate the risky and risk free assets in these two accounts.

There is a large literature addressing the asset allocation issue. In most of the analysis taxes are nonexistent. However, due to certain tax advantages in various tax deferred retirement accounts (TDRAs) as compared to taxable brokerage accounts (TBAs), taxes can have important effects on portfolio choice in real life. In a taxable account, investors have incentives to realize capital losses yet defer capital gains. This incentives imply that investors hold assets that have done well and sell those assets that have done poorly. Over time, following this strategy will lead to a poorly diversified portfolio. Thus, there is a trade off between the utilization of the tax timing option and maintaining a well-diversified portfolio.

The issue of asset location has received less attention in the literature, but has been the subject of several recent papers. With differential tax treatments between the TBAs and the TDRAs, one asset (e.g., Microsoft stock) can now have multiple after-tax payoff structures. Consequently, the presence of tax-advantaged accounts influences the standard asset location problem. Investors not only must choose how much money to put in each asset, but also where those assets should be held. In the last few years, several researchers have tried to address the portfolio location and allocation problem jointly with limited success. Results presented by the researchers are mixed, and models highly stylized and incomplete (see Barber and Odean (2001)). Dammon, Spatt, and Zhang (2003)(hereafter DSZ (2003)), concluded that the optimal location for the taxable bonds is the TDRA, and the TBA is the optimal location for equities given that borrowing is not restricted. Using arbitrage argument, Huang (2000) obtains similar results with deterministic liquidity shocks. The intuition driving their results is that assets with high yields (e.g., bonds) expose investors to larger tax burdens and should therefore be held in the tax deferred account. Moreover, by holding stocks in the TBA, the investor retains the tax timing option and can meet her liquidity needs without any penalty. In contrast to DSZ (2003) and Huang (2000), Shoven (1999) finds that hosting equity mutual funds in their tax-differed accounts may be optimal as opposed to hosting taxable corporate bonds in that location.

The findings of DSZ (2003) and Huang (2000) are at odds with much of the empirical evidence on asset location. Barber and Odean (2001) find mixed holding of stocks and bonds in the taxable accounts of the customers of a retail brokerage. The DSZ (2003) framework is unable to explain the mixed holdings, and the authors call the empirical findings as *asset location puzzle* in DSZ (2003). These empirical and theoretical findings imply that the models being considered are either incomplete or an appreciable number of investors are making suboptimal investment decisions.

In this paper, we revisit the asset location and allocation problem. The main difference in our analysis as compared to DSZ (2003) is that we consider multiple risky assets as opposed to a single risky asset, which is the case in DSZ (2003); which allows us to analyze different correlation structures for the equities. This seemingly simple extension can have important consequences on the optimal portfolio due to the incorporation of the potential of diversification. In particular, it is often the case that investors will choose to put a mix of equities and bonds in both the TBA and the TDRA for viable parameter values. Moreover, one of the key contributions of our paper is the analysis of the relationship between location decision and the correlation structure of the risky assets.

We ask the specific question whether diversification matters in asset location decision. If it does, what is the key determinant of it? How does diversification relate to borrowing constraint? When does diversification outweighs the tax timing option in the TBA leading to mixed holdings in the TDRA?

Why does adding a second risky asset change the optimal portfolio so much? The answer is fairly simple. With a single risky asset, there is no concern about diversification of the risky asset. With multiple risky assets, the desire to defer gains and realize losses leaves an investor with a poorly diversified portfolio. By shifting some equities into the TDRA, the investor can freely trade these assets to maintain a diversified portfolio without incurring the tax consequences. The goal of the investor is to reduce the tax burden on the financial asset holdings as much as possible while at the same time maintaining a portfolio as diversified as possible. In part, these results are also due to more realistic treatment of constraints individual investors face on short sales and borrowing. Moreover, by considering multiple risky assets, we increase the value of the tax timing option, which is understated when a single market index is considered.

When investors are restricted from borrowing and short selling, they are unable to realize the full benefit of their tax timing options. We explore the impact that these restrictions can have on the optimal portfolio. From the casual observation of investment opportunities, institutional regulations, and industry practices it is perceived that borrowing and short selling are not as readily available services for average investors as have been presented in the literature. Moreover, whenever borrowing is allowed, the interest rate of borrowing and lending are not the same. Hence the liquidity needs cannot be met as easily with borrowing or selling of equities.

Cross-basis effects due to holding of multiple risky assets become very important with regards to liquidation for certain shocks. Liquidation may also be motivated by diversification concerns. For example, if a portfolio becomes heavily weighted in one risky asset, in order to maintain a desired level of diversification one may need to liquidate some of her position, and the liquidation decision is not independent of the consideration of the cost basis of other assets in the portfolio. Sometimes portfolio rebalancing for diversification purpose may reduce the value of the tax timing option. To get insight into all these issues we extend the existing models as discussed below and make qualitative conclusions.

To capture the diversification or portfolio rebalancing issues in a more pronounced manner, we consider two risky assets or equities, and a risk free taxable bond in our model. We capture the cross-basis effect by allowing for various parameterizations of the basis of the risky assets. To capture the institutional restrictions such as borrowing and short selling, we consider three different cases, 1) constraints on both borrowing and short sales, 2) constraints on short sales but not on borrowing, and 3) constraints on none. We also consider various correlation structures in order to have specific insight into diversification. Limited analysis of bequest motive, retirement contribution limit, and optimal size of the retirement account is also conducted.

Our model predicts mixed holding of equity and bond in either the TBA or the TDRA under borrowing constraint when the risky assets' prices are independent to each other. The level of the mix is dependent upon the retirement wealth ratio[1]. But mixed holdings are not observed in both accounts simultaneously. This finding is consistent with the DSZ (2003). The mix in the TDRA tends to disappear as borrowing constraint is gradually relaxed. When borrowing constraint is relaxed, we still observe mixed holdings of equity and bonds in the TDRA for high values of retirement wealth ratio. The level of borrowing required to have all bonds in the TDRA is exorbitant in some cases. Correlation structure of the risky assets is a key determinant of the location decision. Negative correlation most of the times lead to mixed holding in the TDRA. Some of these results are sensitive to parameter values such as retirement wealth ratio and cost basis. Moreover, we notice that with multiple risky assets, the asset specific Sharpe ratio may be a key determinant of bond holding in the TBA under borrowing constraint. The bequest motive does not affect asset location and allocation decision in substantial manner during the working age but does affect the investor's consumption and equity holding during the retirement age. Retirement contribution limits do not affect location decisions but do affect allocation decisions in some cases. Furthermore, optimal size of the retirement account is related to borrowing ability.

The rest of the paper proceeds as follows. In section 2, we discuss the existing

[1] Retirement wealth ratio is measured as the fraction of the total wealth that is held in the retirement account. More detailed description is given in the following sections and the technical description is given in the appendix.

models and their results, in section 3 we present the model and the methodology of our investigation, in section 4 we present the results, and we conclude in section 5.

2 Literature Review

As has been noted in Leland (2000), portfolio management entails two decisions. The investor should first specify the target or ideal portfolio strategy, which determines the desired proportions of investment in different classes of assets such as stocks and bonds. The literature provides a rigorous framework for static and dynamic asset allocation in the absence of trading costs and other market frictions to facilitate the first decision (e.g., Markowitz (1952) and Merton (1971)). The second decision is how to implement the desired strategy. Issues to be considered in this implementation step include taxes, borrowing costs, market structure, financial regulations, trading costs, and costs related to other market frictions. Most of the papers in the portfolio optimization area consider the two-decision process separately, abstracting away from the dynamic interaction of the decisions and their outcomes. Our paper addresses both of the decisions and their interactions.

Constantinides (1983) introduced the idea of a tax timing option and pioneered the study of optimal investment and liquidation policy under capital gains taxes. According to the prescription in Constantinides (1983), investors should realize capital losses immediately and defer capital gains realization indefinitely. An important assumption is that there are no restrictions on short sales. In that case, the optimal portfolio choice is separable from the liquidation policy, since "overexposure" to an asset for tax timing reasons can be "undone" by offsetting short positions. A substantial amount of work has built around this basic foundation. Constantinides (1984) shows an application of the findings of Constantinides (1983). Dammon and Spatt (1996) derive the value of tax timing option with the key assumption that investors can circumvent the wash sale rules[2]. For further references on this strand of literature interested readers may consult Dammon and Spatt (1996).

Some works in the existing literature on asset location and allocation focus on an investor who has access to one risky asset and one risk free asset, a TBA and a TDRA. The investor needs to decide in which assets she should invest, and where those assets should be hosted. The tax environment plays a major role in this decision process. Most of the works in the existing literature suggest that bonds should be held in the TDRA and equities should be held in the TBA. If an investor is holding bond in the TBA, she will receive interest payments, which would be taxed at the rate of her ordinary income tax. But if bonds are held in the TDRA, the taxes on interest payments could be deferred. Hence the accumulated interest payments can grow over time much faster in the TDRA than in the TBA. By having equities in the TBA, the investor can get the benefit of the growth in the equity prices without realizing the capital gains but can benefit by realizing capital losses. The following paragraphs explain the intuition of this analysis, using the arbitrage arguments presented by DSZ (2003).

Investors receive dividends from the investments in stocks and interest payments from their investments in bonds. In the TBA, either type of income is taxed at the rate of the investor's personal income tax rate, τ_d, while capital gains are taxed at the

[2] Current IRS rules restricts wash sales. The enforcement and scope of it has increased over time.

capital gains tax rate, τ_g. Suppose that the rate of interest (r) is greater than the dividend yield (d_i) of asset i, so for assets held in the taxable account, the tax burden on the income from bonds is greater than that on the dividend income from stocks ($r\tau_d > d_i\tau_d$). Taxes on capital gains (or losses) on equities in the TBA are due upon liquidation not upon accrual, so investors can avoid paying taxes by continuing to hold the equities. In contrast, the tax on income (interest and dividend payments) or on capital gains from equities can be deferred into the future (e.g., until retirement) if the assets are held in the TDRA. Moreover, due to the basis step up provision, all the taxes may be forgiven at the death of the investor. As a result of these features, the choice of asset location affects the after-tax payoff structure of the assets. These in turn may and affect the asset allocation decision.

Assume that an investor is shifting one of her dollars from stock i to bonds in her tax deferred account, which is offset by a shift of x_i dollars from bonds to that stock in her taxable account. Denote the change in the net cash flow at the end of the year by ΔC_i then we have the following

$$\Delta C_i = r - [(1+\widetilde{g}_i)(1+d_i) - 1] + x_i \{[(1+\widetilde{g}_i)(1+d_i(1-\tau_d)) - \widetilde{g}_i\tau_g - 1] - r(1-\tau_d)\} \quad (1)$$

where $\widetilde{g}_i$ is the random capital gain. Taking the partial derivative of ΔC_i with respect to $\widetilde{g}_i$ and setting it to zero provides an expression, which does not change in value due to any change in $\widetilde{g}_i$. Solving for x_i gives

$$x_i = \frac{1+d_i}{1+d_i(1-\tau_d) - \tau_g} \quad (2)$$

which can then be plugged back into (1), and after rearranging the terms we obtain

$$\Delta C_i = x_i \left[\frac{(r - d_i)(\tau_d - \tau_g)}{1 + d_i} \right] \tag{3}$$

Note that the expression of ΔC_i is free of $\widetilde{g}_i$ and it represents a risk-free, after-tax cash flow that an investor can generate by simple relocation of the assets without incurring any cost. This is the basic intuition of the arbitrage argument.

Given that the term $\tau_d - \tau_g$ in (3) is positive,[3] it is clear that the sign of ΔC_i depends on the sign of the term $r - d_i$. Since the interest on bonds is generally higher than the dividend yield on a stock, this spread is typically positive. When ΔC_i is positive, the investor is strictly better off holding the bond in the tax deferred account and holding stock i in the taxable account. Note that $\frac{\partial \Delta C_i}{\partial d_i} < 0$, hence ΔC_i is monotonically decreasing in d_i. Thus, for every asset where $r - d_i > 0$ the investor allocates the bond in the tax deferred account, and the stocks in the taxable account.

In general, the investor will always prefer to allocate her entire tax deferred wealth to the asset with the highest yield, and allocate other assets in the taxable account. This basic principle can be used to decide on the location of any financial assets including various stock funds, mutual funds, taxable and non-taxable bonds. In DSZ (2003) further results are obtained by imposing restriction on borrowing. With this constraint, the investor first shifts her tax-deferred wealth into the asset with highest yield. Then she allocates her remaining tax-deferred wealth to the asset with the next highest yield, and so on. The process continues with successively lower yielding assets until the investor's tax-deferred wealth is allocated completely. Then she makes offsetting adjustments in the taxable account for the desired level of risk exposure. But sometimes

[3] In the case where $\tau_d = \tau_g$, the location of the assets does not matter.

it may not be possible to make the offsetting adjustments due to the borrowing and short-sale restrictions. This may restrict her from having all bonds in the TDRA. Thus, with borrowing and short sale constraints, the investor may hold a mix of taxable bonds and equities in her tax deferred account, but only if the investor holds all equities in her taxable account.

Shoven and Sialm (2001) also address the asset location and allocation questions but they do so using considerably different modeling techniques. They use a continuous time approach but ignore the consumption/savings decision. But they consider a broader menu of assets, including mutual funds and tax-exempt municipal bonds. They conclude that the preferred asset location is determined primarily by the tax rates facing the asset returns, and assets with the high tax rates should be allocated in the tax-deferred account. Taxable bonds should be held in the tax-deferred account whereas tax-exempt bonds should be held in taxable account. Stocks can be held in either account depending on the tax-efficiency of the stock or stock portfolios.

The prescription of the existing theoretical models for investors to put taxable bonds in the tax deferred account, and equities in the taxable account stands in sharp contrast with observed investor behavior. DSZ (2003) considers this inconsistency as an *asset location puzzle*. Using the data from the Survey of Consumer Finances, Bergstresser and Poterba (2001) show that many households hold stocks in their tax-deferred account but not in their taxable account. This suggests that substantial group of households do not follow a "bond first in the tax-deferred account" asset location strategy. In a similar study, Barber and Odean (2001) find mixed holdings of stocks and bonds in both taxable and tax deferred accounts (data sources are retail and discount brokerages).

One possible explanation for the discrepancy between theoretical predictions and empirical facts are the modeling assumptions. With restrictions on borrowing and short selling, DSZ (2003) predict a mix of stocks and bonds in the TBA only after the TDRA allocation is totally in bonds or vice versa. For most individual investors borrowing and short selling constraints are probably binding. But it is also possible that mixed holdings are motivated by diversification concerns. And mixed holdings may be observed with borrowing and short sale constraints, and without those constraints. Moreover, the retirement wealth ratio of the investor in general matters in portfolio choice.

Amromin (2001) offers an explanation for the empirical findings. His explanation is based on liquidity and the accessibility restrictions on assets in the TDRA. Investors would like to have access to their investment funds in the event of a negative income shock. Due to restrictions on accessing funds in the TDRA, investors have a preference for having some bonds in the TBA.

In this paper, we introduce a dynamic model with multiple risky assets to consider the diversification issue with regards to asset location and allocation decisions. We also consider the impact of various restrictions such as borrowing and short selling constraints on asset location and allocation. To the best of our knowledge, no one has considered the asset location and allocation issue with multiple equity assets yet. Most of the modeling has been done by using only one risky asset (generally interpreted as a well-diversified market equity portfolio) and a risk free bond. The absence of multiple risky assets restricts us from observing the interaction of value of diversificaton and tax timing option. If the goal of the investor is to hold a well-diversified portfolio while achieving as much tax benefit as possible within the tax and institutional constraints then the existing models are certainly limited in many ways. We capture the impact of diversification by having multiple risky assets in our model. We also consider the robustness of our prediction to bequest motives. Our extension helps in explaining some of the inconsistencies between theoretical and empirical findings.

3 The Model

We consider a 20 year old investor who works for 50 years, then lives in retirement for another 30 years before dying at age 100. The investor tries to smoothen her consumption, and wants to leave some of her wealth as a bequest. Here we assume that the investor and her descendant have the same preference over consumption and formulate the bequest as an H-period annuity, i.e. whatever amount the investor leaves for her descendant at her death is invested in an annuity, and the descendant receives the annuity payment over H years.

The investor has access to a TBA, a TDRA, two equities, and a taxable bond maturing in one year. Each equity price follows a binomial price process. The equity price processes are assumed to be independent of each other [4]. During her working years, she receives labor or non-financial income L each period. She is contributes a fixed proportion of her labor income to her TDRA. But she is not allowed to withdraw any amount out of this account till retirement [5]. The investor seeks to maximize her

[4] We consider the correlated cases in latter sections.

[5] We make this simplifying assumption for the ease of our analysis. Moreover, given the penalty

utility of lifetime consumption by choosing seven control variables at each point in time. The control variables are the allocation toward consumption (C_t), the number of shares of equity one and two held after rebalancing the TBA holdings (n_{1t} and n_{2t}), the fraction of retirement wealth held in equities (θ_{1t} and θ_{2t}), and the fraction of retirement wealth held in bonds (θ_{3t}) after rebalancing the TDRA . The rate of inflation is denoted by i.

The formal optimization problem (with the constraints to be discussed later) is as follows[6]

$$\max_{C_t,n_{1t},n_{2t},B_t,\theta_{1t},\theta_{2t},\theta_{3t}} E_0 \left[\sum_{t=0}^{T-1} \beta^t U\left(\frac{C_t}{(1+i)^t}\right) + \beta^T \sum_{j=1}^{H} \beta^j U\left(\frac{A_H \overline{W}_T}{(1+i)^T}\right)\right] \tag{4}$$

The objective function has two components. The first component represents the utility of the investor, defined over real consumption, throughout the working and retirement years (discounted at the rate of β). The second component represents the utility of the descendant out of the consumption of the bequest over H years (discounted at a rate of β). The ending period wealth is $\overline{W}_T$ and A_H is the H-period annuity factor. The expectation is taken over the whole expression due to the random processes followed by the equity prices.

The maximization of the objective function, eq. (5), is subject to a number of constraints. We describe these conceptually here and refer the reader to the Appendix for a formal presentation [7].

one has to pay in order to withdraw money from the tax deferred account before retirement, it is unlikely that an average investor would want to do that in regular circumstances.

[6]To maintain the clarity of the exposition, we describe most of the notations and definitions in the appendix.

[7]In the Appendix, the constraints C1 - C9 is presented in order as Equation 6 - 14.

C1 Total wealth each period is the sum of the wealth in the the TBA (see C2) and the TDRA (see C3) after subtracting the tax liability.

C2 Wealth in the TBA each period is the after tax-labor income plus the value of the holdings of the equities and bond after paying taxes on interest and dividends.

C3 Wealth in the TDRA each period is the prior-period ending balance times the TDRA portfolio gross return (no taxes are deducted).

C4 Consumption each period is the residual left after subtracting from total wealth (C1) the value of both investment accounts and any capital gains taxes in the TBA.

C5 The amount of money in the TDRA at the start of each period until retirement is the prior balance plus a fraction (α) of the labor income.

C6 The amount of money in the TDRA at the start of each period in retirement is the prior balance less a withdrawal used for subsistence.

C7 Consumption must be non-negative in every period. Short selling is not allowed in the TBA [8]. Borrowing is not allowed in the [9]. And investors hold non-negative amount of each asset in the TDRA.

C8 The investor must liquidate positions at death.

C9 Average cost basis is used.

[8]We relax this constraint latter on to analyze the consequences of having this constraint.

[9]We relax this constraint latter on to analyze the consequences of having this constraint.

We use the power form of the utility function with constant relative risk aversion coefficient $\gamma > 0$.

$$U(C_t) = \begin{cases} \ln(C_t) & \text{if } \gamma = 1 \\ \frac{C_t^{1-\gamma}}{1-\gamma} & \text{otherwise} \end{cases}$$

This form of the utility function is chosen due to the convenience of being able to make conclusions that are independent of wealth level, and some empirical works supports the viability of this form. This allows us to ignore wealth as a state variable, which simplifies the analysis.

In the dynamic optimization problem, the investor observes a vector of state variables $X_t = \left[P_{1t}, P^*_{1t-1}, P_{2t}, P^*_{2t-1}, n_{1t-1}, n_{2t-1}, W_t, Y_t, L_t\right]$. These variables contain information known at the time the investor chooses the control variables. As a modeling convenience, we follow DSZ (2003) and normalize the vector of state variables to obtain $x_t = [s_{1t}, s_{2t}, p^*_{1t-1}, p^*_{2t-1}, y_t]$ [10]. The symbols s_{kt}, p^*_{kt-1}, and y_t represent prior holding of asset k, $k = 1, 2$, as a fraction of wealth in the TBA, W_t, basis-price ratio at time t, and wealth in the TDRA as a fraction of the beginning of the period total wealth before trading at time t, respectively. The consumption wealth ratio is c_t [11]. The variables b_t, f_{1t}, and f_{2t} track the fraction of wealth in the the TBA allocated to risk-free taxable bond, equity 1, and equity 2, respectively. The fraction of the TDRA wealth allocated to equity 1, equity 2, and the taxable bond are θ_{1t}, θ_{2t}, and θ_{3t}, respectively. We assume that labor income or non-financial income is a constant fraction, l, of the total wealth for all t. During retirement age labor income or non-financial income is

[10]This allows transformation of the optimization problem into a more tractable form. Please consult the Appendix for normalization procedure, and the reformulated optimization problem.

[11]c_t is the fraction of the total wealth allocated for consumption at time t.

assumed to be zero.

3.1 Limitations and Potential Extensions

Certain limitations of the model need to be recognized. In our model, we have deterministic labor or non-financial income i.e. it does not follow any random process of its own. It constitutes a fixed portion of the total wealth. This assumption allows us to use the homogeneity of the objective function in reducing the dynamic optimization problem (we discuss this in the Appendix). Given the findings of Amromin (2001) and Huang (2000), we believe that the labor income shock resulting in a liquidity shock will bring in more validity to our conclusion as opposed to questioning its robustness. We use average cost basis calculation rather than exact cost basis. In an exact cost basis calculation, the changing path of the cost basis is taken into account for each transaction for a given asset, and cost basis is updated accordingly. The loss due to this simplification should be minor. DeMiguel and Uppal (2003) reports that the certainty equivalent loss from using the average tax basis instead of exact tax basis is less than 0.2% for problems with five periods and is less than 0.5% for problems with ten periods. Even though the loss is increasing in the time periods considered, the impact of it would be minor on our qualitative conclusions due to the magnitude of the optimized values of the relevant variables that we use to make our conclusions. For a much richer analysis, including specific stochastic process for labor income and exact cost basis would be desirable. Moreover, including stochastic arrival of death would allow us to observe its impact on the decision making process in different parts of the investor's life. But we suspect that the overall asset location and allocation pattern would not change in any drastic manner.

4 Results from Numerical Optimization

We solve the investor's dynamic optimization problem numerically. The Appendix contains some technical details and a summary follows below. The life span is deterministic for the investor. The sources of uncertainty are the equity price processes. We assume that each of the equity follow a binomial process. The investor is allowed to have different bequest motives which we capture by varying the parameter H.

To parameterize the model, we make the following assumptions[12]. The agent lives for 100 years, she starts working at age 20. For our optimization we consider the 80 years. The investor works till the end of the first 49 years and then retires. The equities have mean returns, μ_n, where $n = 1, 2$, of 9% and 13%, respectively. Corresponding standard deviations, σ_n, where $n = 1, 2,$, are 20% and 30%. Both stocks pay a 2% dividend and the bond pays interest of 6%. The tax rates are 36% for ordinary income and 20% for capital gains. The inflation rate is 3.5%.

The risk aversion parameter, γ, is set at 3. We set the fraction of total wealth earned as labor or non-financial income, l, at 15% at any given time during the working age. The portion of labor income saved in the retirement account, α, is 20%. To capture the bequest motives we vary the bequest horizon parameter H. But for most of the analysis, we set H at 20. In other words, in most of the cases we analyze an investor who intends to leave a bequest amount that would allow her heir to have a fixed level

[12]We closely follow the parameter values from DSZ (2003) to make the comparison of the qualitative conclusions easier. DSZ (2003) documents viable reasons for selecting certain parameter values.

of consumption for 20 years. The preferences of both the investor and her heir over consumption are assumed to be the same.

The numerical optimization is done over a grid of possible values for the state vectors. We created a $5 \times 5 \times 5 \times 5 \times 8$ grid for the state space and used this grid for each of the 80 periods for optimization where $s_{1t} \in [0.01, 1]$, $s_{2t} \in [0.01, 1]$, $p^*_{1t-1} \in [0.01, 1.2]$, $p^*_{2t-1} \in [0.01, 1.2]$, and $y_t \in [0.1, .8]$. We have tried different parameterizations within the stated range but we report only a few interesting parametric outcomes. We solve the dynamic optimization problem by using backward induction and linear interpolation.

In the figures and tables, low value of parameters s_{1t} and s_{2t} (e.g., 0.1) indicate lower level of prior equity holdings, and high value of parameters s_{1t} and s_{2t} (e.g., 0.7) indicate higher level of prior equity holdings. If the basis-price ratio is above one ($p^*_{1t-1} > 1$ or $p^*_{2t-1} > 1$), we have built in capital losses. When the basis-price ratio is below one ($p^*_{1t-1} < 1$ or $p^*_{2t-1} < 1$), we have a built in capital gains, and when it is equal to one ($p^*_{1t-1} = 1$ or $p^*_{2t-1} = 1$) there are no capital gains or losses.

Here we focus specifically on parameter values that lead to interesting results. Since our goal is to address the limitations of the existing models and extend the models, we try to recognize the potential limitations of the existing models as much as possible. Existing theoretical findings fail to explain the mixed holdings observed in the empirical data and ascribe this to suboptimal decision making in the part of the investors. Here we present our work in a way that may help us in understanding some of the reasons for these theoretical inconsistencies. The results in this section are organized as follows. First, we discuss the investor's consumption choices. Second, we turn to the asset location and allocation decisions. Then we consider the issue of bequest motives. We discuss the retirement contribution limit, and its impact on location and allocation decision in the following subsection. In the last part of this section, we discuss the optimal size of the retirement account. For the ease of presentation, we start with a base case portfolio and vary parameter values of the base case in order to get other specific portfolios. For the base case we consider a portfolio that is heavily weighted in equities, both equities are equally weighted, and there are no built in capital gains or losses, ($s_{1t} = .4$, $p^*_{1t-1} = 1$, $s_{2t} = .4$, $p^*_{2t-1} = 1$), please refer to Table 1 for details. For each of these portfolios, we examine the results with and without constraints on borrowing and short selling. We report combined equity allocation throughout this presentation.

4.1 Consumption

Consumption is relatively smooth across age and income levels. The level of consumption varies according to age, bequest motive, level of retirement wealth, correlation structure of the risky assets, and restrictions on trading. Depending on various parameter values, the level of consumption, c_t, expressed as the fraction of the total wealth allocated toward consumption may vary between $3.5 - 13\%$. Our findings regarding consumption is mostly consistent with the existing findings in the qualitative sense.

We simulate for three different cases due to trading restrictions. In the first case we don't allow any borrowing and short selling, in the second case we allow borrowing, and in the third case we allow both borrowing and short selling. When borrowing and short selling are allowed the investment opportunity set for the investors expands which allows the investors to consume at a higher level. We present the impact of the restrictions graphically for the investors with average retirement wealth ratio, $y = .4$, in Figure 1(a). In this figure, $C1$ represents the level of consumption with restrictions on both borrowing and short selling, $C2$ represents the level of consumption when only borrowing is allowed, and curve $C3$ represents the level of consumption under no restrictions. Level of consumption is lowest in $C1$ but higher in $C2$ and $C3$. Consumption is decreasing in age irrespective of trading restrictions. Notice further that the gain in consumption from the relaxation of short sale constraint is not as substantive as it is in the case of borrowing constraint relaxation. The sharp decline in c_t during the working age is due to the fixed contribution that the investors make towards the TDRA out of their non-financial income. This fixed contribution increases the level of the total wealth of the investors over time. Since consumption level is relatively constant and the total wealth is growing over time, the value of c_t declines sharply during the working years. But the decline in c_t during the retirement age is relatively minor due to the absence of the fixed contribution towards the TDRA. Rather then contributing to the TDRA, during the retirement age investors withdraw wealth from the TDRA for consumption purposes. The investors try to balance their consumption with their bequest contributions. But c_t increases dramatically for the investors with low bequest motive. Since the investors are not leaving much for their heirs, they spend more of their total wealth for consumption. This intuition is graphically presented in Figure 4(b).

In general, consumption is decreasing in age and mildly increasing in retirement wealth ratio. Figure 1(b) presents consumption across retirement wealth ratio over time under borrowing constraint. Here consumption is smooth across time and retirement wealth ratio. When borrowing constraint is relaxed the consumption surface shifts upward but the relative consumption patterns are maintained for all retirement wealth ratios (see Figure 1(c)) . Retirement wealth level affects consumption level but the impact is minor, in Figure 1(d) we consider two retirement wealth levels to make this point. Here we consider investors who are not allowed to borrow. In Figure 1(d), we present consumption profiles of two investors. The first investor has low retirement wealth ratio, $y = .1$, and the second investor has high retirement wealth ratio, $y = .8$. As can be observed from the figure, consumption level is higher for the investor with higher retirement wealth ratio.

Within any single period, consumption is relatively smooth. In any single period consumption level is dependent on the basis-price ratios of the equities and equity holdings. If equity holding level for either equities is high, the determinant of the consumption level is the basis-price ratio. If the basis-price ratio is low i.e. there is substantial built in capital gain then the consumption level would be low. But if there is built in capital loss consumption level may be higher. If there are built in capital losses, in the beginning of the period the investors will capitalize on the tax timing option i.e. they would claim the capital losses, and either consume more or rebalance their portfolios with the proceeds. Further, if prior equity holding is high consumption level is higher in basis-price ratio. The consumption level of an investor with high equity holding with built in capital gains is relatively low compare to the investor with low equity holding with built in capital gains. The fraction of wealth that an investor with high level of wealth, due to built in capital gains, would spend for consumption would certainly be smaller because of their high net worth. But this outcome changes when basis-price ratio is greater than one.

4.2 Asset location and allocation

We observe interesting mix of bonds and equities in both of the accounts, the TBA and the TDRA, with and without constraints. The location and allocation decisions are sensitive to specific parameter values such as the basis-price ratios of the equities, the retirement wealth levels, Sharpe ratios of the equities, correlation structure of the equities, and the trading restrictions. We present the results in the following way. We first present asset location and allocation decisions for the base case portfolio under various constraints. Then we present the asset location scenarios under various correlation structures. The relationship between retirement wealth ratio, y, and the TDRA equity holding under various levels of borrowing restriction is discussed at the end of this section. In reporting equity allocations, we report the combined allocation of both of the equities.

Consider the base case portfolio described earlier, see Table 1 for the set of parameter values. In this scenario, with borrowing and short sale constraints investors tend to hold almost no bond in the TBA irrespective of the retirement wealth ratio, see Figure 2(c). Here equity allocation in the TBA is decreasing in retirement wealth ratio during the working years. The allocation to equity varies between $100 - 40\%$. But equity holding is increasing in the retirement wealth ratio during the retirement age, see Figure 2 (a). For investors with high retirement wealth ratios, it is optimal to allocate most of their funds to equities in the TDRA for tax deferred growth whereas the allocation in the TBA is more suitable for immediate liquidity needs. Holding equities in the TBA allows them to meet their liquidity needs, capture the growth in capital gains, and leave the door open for exercising the tax timing option. As opposed to investors with higher retirement wealth ratios, for investors with lower retirement wealth ratios liquidity needs may be much stronger. So they tend to hold more stock in the TBA, and try to meet their liquidity needs as well as try to capitalize on the tax timing option. But during retirement age bequest motive pushes the equity holding further up. This push may be due to the potential of capitalizing on tax basis step up or due to the availability of retirement funds that allows the investor to spare more for investments. Notice in Figure 2(a) for the investors with high retirement wealth ratios, the allocation towards equities during retirement years may go above 100%. The allocation is measured as a fraction of the wealth in the TBA at any given time. During the retirement age, the non-financial or labor income is zero but the investor receives returns from investments in the TBA, and a fraction of the TDRA wealth is also available from the retirement distributions. Given the bequest motive and reduction in gain from tax deferral during retirement, investors with high retirement wealth ratios may invest the extra money they receive from the TDRA distributions (amount available after consumption) in the TBA. Hence the relative investment amount in equity in the TBA may go beyond 100% even with the borrowing constraint individuals with high y. The proportion of overall equity holding in the TBA tend to remain the same but the composition of the portfolio is affected by the tax basis and prior equity holdings. Given the opportunity of diversification, investors may opt for more equities given better risk adjusted return.

Sometimes location of bond in the TBA may be motivated by the inability to borrow to meet the liquidity needs. We notice an interesting relationship between the Sharpe ratio and bond holding. Up to certain threshold Sharpe ratio, bond allocation in the TBA is negatively related to the Sharpe ratio. We present this relation graphically in Figure 4(a).

In the TDRA, bond holding is decreasing in retirement wealth ratio across time. For low retirement wealth ratio almost 100% of the retirement wealth is allocated to bond. But it declines substantially over retirement wealth ratio, and it is below 50% for investors with high retirement wealth ratios, see Figure 2 (d). The equity allocation is increasing in retirement wealth ratio in the TDRA, see Figure 2(b). Here we obtain results similar to DSZ (2003). We have only equity allocation in the TBA and mix allocation in the TDRA. But mixed holdings in both accounts are not observed simultaneously.

When borrowing is not restricted, the investors hold no bonds in the TBA irrespective of their retirement wealth ratios [13]. Borrowing level is low for the investors with low retirement wealth ratios but investors with high retirement wealth ratios borrow heavily, see Figure 3(c). The relaxed (limited) borrowing constraint is binding for investors with higher retirement wealth, $y = .5 - .8$. Notice that for investors with lower retirement wealth, $y = .1 - .5$, the constraint is not binding. Even though these investors could borrow more, they do not borrow to their maximum capacity. The fourth column of Table 2 and Table 3 captures this for investors with parameter values

[13] Throughout the presentation when we mention borrowing is not restricted we mean limited relaxation of the borrowing constraint. For the base case analysis, we allow the investor to borrow up to 100% of her wealth in the TBA. Unless otherwise stated, whenever we mention that there is no borrowing constraint, it should be understood that we are allowing the investor to borrow 100%.

$y = .1$ and $y = .8$. The mean borrowing levels over lifetime are 14% and 100% for investors with low and high retirement wealth ratios, respectively. In the absence of borrowing constraint, positive amount of bonds is held in the TDRA most of the times. Investors with base case portfolio with low or average retirement wealth ratio would hold 100% of their retirement wealth in bonds 100% of the times over their entire lifetime, see column 5 in Table 3. But bond holding declines for investors with higher y. The reduction in bond holding is almost 50% for this type of investors, see Figure 3(d).

When the borrowing constraint is relaxed, investors borrow heavily to capitalize on higher returns on the equities. The equity holding shifts upward, and the equity holding may range from $100 - 250\%$ of the wealth in the TBA depending on y and age. But the allocation pattern during working age and retirement age remains very similar to the no constrained case, see Figure 3(a) for the base case. Given the relaxed borrowing constraint, investors with high y borrow heavily in the TBA to capitalize on higher returns from the equities. They hold equities in both portfolios. The split between equity and bond holding in the TDRA is almost 50:50. But the investors with lower level of retirement wealth are not as aggressive in borrowing (for them the borrowing constraint is not binding), they borrow money to hold more equity in the TBA but do not hold any equity in the TDRA. Their main allocation of funds is in bonds. By relaxing short sale constraints we obtain similar qualitative conclusions.

Now we consider various correlation structures of the equities. We consider equities with correlation coefficients (ρ) of -1, -0.6, 0, 0.6, and 1. When the equities are perfectly positively correlated ($\rho = 1$) the equities do not serve much terms of diversification. But close to 60% of the wealth in the TBA is held in equities with and without borrowing constraint to capitalize on the risk adjusted return and tax timing option. But as the magnitude of the correlation declines and the sign becomes negative the equity holding increases in the TBA with and without borrowing constraint. Interesting insights are obtained with regards to TDRA. The location decision is influenced by the correlation structure of the equities. With negative correlation the value of diversification is high so investors always hold 100% in equities in the TDRA. But if $\rho = 0$ i.e. if the equity price processes are independent of each other, the equity holding is higher with borrowing constraints, see Figure 5(a) and (b). When investors are allowed to borrow as much as 100% of their brokerage accounts' wealth, they hold no equities in the TDRA if the correlation coefficient is positive (if $\rho = 0.6$ or $\rho = 1$). But if the correlation coefficient is negative all the retirement wealth is allocated towards equities, see Figure 5(c) and (d). By relaxing the borrowing constraint further it is possible to have all in bonds in the TDRA but the level of borrowing is too high to be feasible. But when the equities are perfectly negatively correlated ($\rho = -1$), it is not possible to have bonds in the TDRA even with high level of borrowing. The diversification benefit is simply too high. The diversification value simply outweighs the tax timing option value.

One of the key determinants of bond location in the TDRA is the borrowing constraint. We document the interaction of retirement wealth ratio (y) and borrowing constraints in determining the bond holding in the TDRA for some specific parameter values. In panel (a) and (b) of Figure 6, we consider two scenarios. In panel (a), we have an investor with $y = 0.4$. This investor holds a mix of bonds and equities in the

TDRA. But when he is allowed to borrow 100% of her TBA wealth, she holds all bonds in the TDRA. For making the point more clear we also have an intermediate case where we allow the investor to borrow only 25% of her TBA wealth. This shows the investors location preference for bonds if borrowing is allowed. But the level of borrowing that is required to have all in bonds in the TDRA depends on the retirement wealth ratio this can be observed in panel (b) where $y = 0.7$. Here the investor need to have much more borrowing ability in order to have total preference for bond in the TDRA. The investor needs to be allowed to borrow 175% of her TBA wealth to have all in bonds in her TDRA, which may deem unreasonable given the institutional restrictions. We have tried other scenarios where one equity got appreciated and other lost value; even in those scenarios we obtain similar results.

4.3 Bequest motive

The observations made above are mostly robust to changes in bequest motive, especially during the working years. The most noticeable difference that we observe is in the allocation for consumption, and equity holding. We consider three specifications for this purpose with $H = 5, 20$, and 25 without any borrowing constraint. The consumption level is around 10% of the total wealth during most of the working age but in the retirement age the consumption allocation changes. For low bequest motive, $H = 5$, we observe consumption increases dramatically over time whereas for higher bequest motive, $H = 20$ consumption is decreasing over time (see Figure 4(b)). The growth in consumption for low bequest motive, $H = 5$, is pronounced in retirement age. Even with low bequest motive consumption allocation seems to be within the range of $10 - 15\%$ of the total wealth. As the bequest motive increases consumption in the retirement age decreases sharply with age. When bequest motive is low, the investor derive utility mostly out of her own consumption, hence she makes her consumption allocation decision accordingly. These conclusions seem to hold for variations in initial equity holdings. The magnitude of various allocations vary over bequest motives but the location decisions are qualitatively very similar in the working age. But variations in allocation strategy do arise in the retirement age. To provide some intuition, we consider the equity and bond allocation in the TBA for an investor with average retirement wealth, $y = .4$, holding the base portfolio. The allocations are presented in graphically in panel (c) and (d) of Figure 4. In panel (c), the equity allocation in the TBA is considered. Equity holding in the retirement age declines for low bequest motive but increases for higher bequest motive. An investor with low bequest motive is focused in maximizing her own consumption within her life, so the value of tax timing option or benefit of the step up clause are not as relevant to her. But for a person with higher bequest motive all these issues are important. Holding equity allows her to capitalize on the tax timing option, and on the potential growth in capital gains. Panel (d) presents the bond allocation, and the bond allocation is uniform for all bequest motives. Heavy borrowing in the working age and reduction in borrowing in the retirement age. Investors tend to borrow heavily to capture the benefit of the higher returns of equity and on the deferral option. But in the retirement age investors start to dissave and borrowing is not as much needed to meet the liquidity need for consumption, or asset purchases. Equity holding is almost zero in the TDRA. All the wealth in the TDRA is allocated toward bond for growth with tax deferral.

4.4 Retirement contribution limit

Retirement contribution limit does not affect the asset location decision but within the accounts it affects the allocation decisions. In the base case, retirement contribution level, α, is set at 20%. In order to get insight into the issue we keep all the parameter values for the base case the same except for α. We vary the value of α to observe its impact in location and allocation decision. With borrowing constraint, we observe minor reallocation of assets in the TBA, and all the changes in allocation take place during the working years. With higher value of α investors tend to reduce their equity holding in the TBA by $0 - 5\%$. Investors with the lower y values benefit more from the relaxation of the contribution limit. But the gain in utility from the increase in the value of α is not strictly monotonically increasing, see Table 4. In column two of Table 4, we report the percentage changes in the value of the value function due to the increase in the retirement contribution level from 5% to 20%. We observe that the investor with $y = 0.1$ benefits most and the investor with $y = 0.7$ benefits least from the increase in α. We do a similar analysis without borrowing constraint in this case, the order of benefit received is maintained but the magnitude of the benefit increases, see column three of Table 4. Similar results are reported for the increase in α from 20% to 30% in column four and five of Table 4.

4.5 Optimal size of the retirement account

In our analysis, we have the retirement wealth ratio, y, as a state variable. This can be thought of as a proxy for the size of the retirement account. Any investor would want to smoothen her consumption by allocating her entire wealth in the two accounts optimally. Here we present our limited analysis on this issue. We consider the base case parameters, and vary the y values and try to observe the impact of these variations on the value of the value function or utility. In panel (c) of Figure 6, we present the case when borrowing is not allowed. The investors with higher y values are better off during the retirement years which is quite intuitive. But the interesting case is presented in panel (d) where the investor is allowed to borrow (100%). We observe that optimality is not linearly related to the size of the retirement wealth ratio. Here the investor with $y = 0.6$ is better off than the investor with $y = 0.8$. So it is interesting to notice that there may be a key role of borrowing constraint in the determination of the optimal size of the retirement account.

5 Conclusion

Most of the modeling framework in the existing literature of asset location and allocation promote the view that by having larger coupon rates compare to dividend yield rates an investor is much more exposed to tax burden by holding bonds in the taxable investment account as opposed to holding it in the tax differed account. Moreover, by holding stocks in the taxable account an investor can capitalize on the tax timing option, and meet her liquidity needs without any penalty. But the limitations of the existing literature are substantial that begs careful reformulation and augmentation. In this paper, we introduce the issue of diversification in this discussion. And we find that correlation structure of the risky assets in the portfolio can be a key determinant of location decision. In many cases, we obtain results very similar to existing results.

Specifically, we observe that DSZ (2003) conclusions under borrowing constraints do hold for viable parameter values with multiple risky assets. But we observe that when investors are allowed to borrow the existing conclusions would hold in some cases only under demanding assumptions on borrowing. Moreover, when the risky assets are perfectly negatively correlated hardly any assumption on borrowing would buttress the conclusions in DSZ (2003). We find interesting mix of equities and bonds in the TBA and in the TDRA with and without constraints, which is consistent with the empirical findings. Our analysis of bequest motives show that during the working years the asset location and allocation decisions are quite robust to bequest motives. But we do observe some variations during the retirement age. So in general, we can conclude that an average investor should take into account of the correlation structure of the risky assets in the portfolio along with borrowing constraints, retirement wealth ratio, basis-price ratio, and Sharpe ratios of the risky assets in making asset location and allocation decision or in rebalancing the portfolio. Retirement contribution limit may not affect location decision but it may affect the allocation decision to some extent. We have touched upon the optimal size of the retirement account, but we believe a more thorough investigation would be fruitful in this area. Our limited analysis indicates that there may be an important relationship between the optimal size of the retirement account and the investor's borrowing ability. Addressing some of the limitations and concerns that we have mentioned in the main body of the paper would certainly facilitate our understanding of asset location and allocation further. Most of the existing empirical data is aggregate level data. But more primary, and individual investor level data would certainly strengthen the empirical conclusions, and would allow us to test the quality of the theoretical models. For now our conclusions and analysis suffices as evidence in reconciling some of the concerns regarding the *asset location puzzle*.

6 Appendix

The investor solves the following optimization problem for asset location and allocation.

$$\max_{C_t, n_{1t}, n_{2t}, B_t, \theta_{1t}, \theta_{2t}, \theta_{3t}} E_0 \left\{ \sum_{t=0}^{T-1} \beta^t U(\frac{C_t}{(1+i)^t}) + \beta^T \sum_{j=1}^{H} \beta^j U(\frac{A_H \overline{W_T}}{(1+i)^T}) \right\} \quad (5)$$

such that

$$\overline{W_t} = W_t + Y_t(1-\tau_d), t = 0, \ldots, T \quad (6)$$

$$W_t = L_t(1-\tau_d) + \sum_{k=1}^{2} n_{kt-1}[1+(1-\tau_d)d_k]P_{kt} + B_{t-1}[1+(1-\tau_d)r], t = 0, \ldots, T \quad (7)$$

$$Y_t = W_{t-1}^r \left[\sum_{k=1}^{2} \theta_{kt-1}(1+g_{kt})(1+d_k) + \theta_{3t-1}(1+r) \right], t = 0, \ldots, T \quad (8)$$

$$C_t = \overline{W_t} - \tau_g \sum_{k=1}^{2} G_{kt} - \sum_{k=1}^{2} n_{kt} P_{kt} - B_t - W_t^r(1-\tau_d), t = 0, \ldots, T-1 \quad (9)$$

$$W_t^r = Y_t + \alpha L_t, t = 0, \ldots, Tr - 1 \quad (10)$$

$$W_t^r = Y_t(1-h_t), t = Tr, \ldots, T-1 \quad (11)$$

$$C_t \geq 0, n_{kt} \geq 0, B_{kt} \geq 0, 0 \leq \theta_{it} \leq 1, t = 0, \ldots, T-1, i = 1, 2, 3. \tag{12}$$

$$n_{kT} = 0, B_T = 0, W_T^r = 0 \tag{13}$$

$$P_{kt}^* = \begin{cases} \frac{n_{kt-1} P_{kt-1}^* + max(n_{kt} - n_{kt-1}, 0) P_{kt}}{n_{kt-1} + max(n_{kt} - n_{kt-1}, 0)} & \text{if} \quad P_{kt-1}^* < P_{kt} \\ P_{kt} & \text{if} \quad P_{kt-1} \geq P_{kt} \end{cases} \tag{14}$$

$$G_{kt} = \left\{ I(P_{kt-1}^* > P_{kt}) n_{kt-1} + [1 - I(P_{kt-1}^* > P_{kt})] max(n_{kt-1} - n_{kt}, 0) \right\} (P_{kt} - P_{kt-1}^*) \tag{15}$$

Notations:

C_t= nominal consumption in period t

B_t= amount invested in bonds in the TBA

W_t= wealth in taxable account after payment of the ordinary income taxes but prior to the payment of capital gains taxes at time t

$\overline{W_t}$ = total wealth at time t

W_t^r= wealth in the TDRA after contribution or withdrawal at time t

Y_t= pretax wealth in the tax-deferred account before contribution or withdrawal in period t

L_t= pretax non-financial or labor income at time t

αL_t= contribution to the retirement account from the pretax non-financial income at time t

$h_t Y_t$ = withdrawal from the retirement account at time t

P_{kt} = price of stock k at time t

n_{kt} = number of the shares of stock k held in the TBA

r = nominal risk free interest rate

d = nominal dividend yield

g_{kt} = nominal pre-tax capital gain return from stock k at time t

G_{kt} = total realized capital gain from stock k at time t

τ_d = income tax rate

τ_g = capital gains tax rate

H = number of years for which the investor wants to leave funds for her descendant or bequest horizon

i = inflation rate

Inflation adjusted annuity factor:

$$A_H = \frac{r^*(1+r^*)^H}{(1+r^*)^H - 1}$$

$$r^* = [(1-\tau_d)r - i]/(1+i)$$

We reformulate the constrained optimization problem as a dynamic optimization problem. Then the value function of the dynamic optimization problem at time t, $V_t(X_t)$, is a function of the vector of state variables, X_t, at time t.

$$V_t(X_t) = \max\left\{U(\frac{C_t}{(1+i)^t}) + \beta E_t[V_{t+1}(X_{t+1})]\right\}$$

which is subject to constraints listed in Equations 6 - 14.

$X_t = \left[P_{1t}, P^*_{1t-1}, P_{2t}, P^*_{2t-1}, n_{1t-1}, n_{2t-1}, W_t, Y_t, L_t\right]$

We normalize the value function and some of the new variables to simplify the problem further. Let $v_t = \frac{V(X_t)}{[\overline{W}/(1+i)^t]^{1-\gamma}}$, $l = \frac{Lt}{W_t}$, $s_{1t} = \frac{n_{1t-1}P_{1t}}{W_t}$, $s_{2t} = \frac{n_{2t-1}P_{2t}}{W_t}$, $f_{1t} = \frac{n_{1t}P_{1t}}{W_t}$, $f_{2t} = \frac{n_{2t}P_{2t}}{W_t}$, $b_t = \frac{B_t}{W_t}$, $c_t = \frac{C_t}{\overline{W}}$, $y_t = \frac{Y_t(1-\tau_d)}{\overline{W}}$, $w_t^r = \frac{W_t^r}{\overline{W}}$, $\delta_{kt} = G_{kt}/W_t$, and $p_{kt-1}^* = P_{kt-1}^*/P_t$.

After the normalization, we deal with five state variables, beginning of the period equity 1 proportion or holding in the TBA (s_{1t}), beginning of the period equity 2 proportion or holding in the TBA (s_{2t}), basis-price ratio of equity 1 (p_{1t-1}^*), basis-price ratio of equity 2 (p_{2t-1}^*), and retirement wealth ratio (y_t) at time t, respectively. We redefine the vector of state variables as $x_t = [s_{1t}, s_{2t}, p_{1t-1}^*, p_{2t-1}^*, y_t]$. Note that l denotes the constant fraction of the total wealth that comes from the labor or non-financial income, L_t. Retirement withdrawal rate is denoted by h_t, and it is calculated by using the life expectancy table provided by IRS (h_t is the inverse of life expectancy at time t).

For control variables, we have c_t, b_t, f_{1t}, f_{2t},θ_{1t}, θ_{2t}, and θ_{3t}, consumption-wealth ratio, fraction of taxable wealth allocated to bonds in the TBA after trading, fraction of taxable wealth allocated to equity 1 in the TBA after trading, fraction of taxable wealth allocated to equity 2 in the TBA after trading, fraction of tax-deferred wealth allocated to equity 1 in the TDRA, fraction of tax-deferred wealth allocated to equity 2 in the TDRA, and fraction of tax-deferred wealth allocated to bond in the TDRA, respectively at time t. Using the new notations, we can reformulate the constraints as follows.

$$c_t = 1 - \tau_g \delta_t (1 - y_t) - (1 - y_t) \left\{ b_t + \sum_{k=1}^{2} f_{kt} \right\} - w_t^r (1 - \tau_d) \tag{16}$$

$$R_{t+1} = \frac{b_t[1 + (1 - \tau_d) r] + \sum_{k=1}^{2} f_{kt}[1 + (1 - \tau_d) d_k](1 + g_{kt+1})}{b_t + \sum_{k=1}^{2} f_{kt}} \tag{17}$$

$$R_{t+1}^r = \theta_{3t}(1 + r) + \sum_{k=1}^{2} \theta_{kt}(1 + d_k)(1 + g_{kt+1}) \tag{18}$$

Equations (17) and (18) can be considered as the gross returns in the TBA and the TDRA. With the new notations and reorganization, we have the following reduced dynamic optimization problem

$$v_t(x_t) = \max \left\{ U\left(\frac{c_t}{(1+i)^t}\right) + \beta E_t[v_{t+1}(x_{t+1})] \right\} \tag{19}$$

$$= \max \left\{ \frac{c_t^{1-\gamma}}{(1+i)^t} + \beta E_t[v_{t+1}(x_{t+1}) \overline{w_{t+1}}^{1-\gamma}] \right\} \tag{20}$$

such that

$$\overline{w_{t+1}} = \left(\frac{R_{t+1}}{1+i} \right) (b_t + \sum_{k=1}^{2} f_{kt}) \frac{1 - y_t}{1 - l(1 - \tau_d)} + \left(\frac{R_{t+1}^r}{1+i} \right) \frac{w_t^r (1 - \tau_d)}{1 - l(1 - \tau_d)}, t = 0, \cdots, T - 1 \tag{21}$$

$$w_t^r = \frac{y_t}{1 - \tau_d} + \alpha l, t = 0, \cdots, Tr - 1 \tag{22}$$

$$w_t^r = \frac{y_t}{1 - \tau_d}(1 - h_t), t = Tr, \cdots, T - 1 \tag{23}$$

$$c_t \geq 0, f_{kt} \geq 0, 1 \geq \theta_{it} \geq 0, i = 1, 2 \tag{24}$$

At the terminal date T the value function take the value v_T.

$$v_T = \frac{\beta(1-\beta^H)A_H^{1-\gamma}}{(1-\beta)(1-\gamma)} \tag{25}$$

We create a $5 \times 5 \times 5 \times 5 \times 8$ grid for the state space and use this grid for each of the 80 periods for optimization where $s_{1t} \in [0.01, 1]$, $s_{2t} \in [0.01, 1]$, $p^*_{1t-1} \in [0.01, 1.2]$, $p^*_{2t-1} \in [0.01, 1.2]$, and $y_t \in [0.1, .8]$. We solve the dynamic optimization problem by using backward induction and linear interpolation. We keep the state space relatively small to economize in computation time and difficulty. But grid points are selected in such a way that allows us to obtain qualitative inferences about the location allocation decisions for most of the cases.

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Table 1: Table of parameter values for the base case portfolio

Parameters	Base portfolio
Basis-price ratio of asset 1 (p^*_{1t-1})	1.0
Basis-price ratio of asset 2 (p^*_{2t-1})	1.0
Prior holdings in equity 1 (s_{1t})	0.4
Prior holdings in equity 2 (s_{2t})	0.4
Mean return of equity 1 (μ_1)	9%
Standard deviation of equity 1 (σ_1)	20%
Mean return of equity 2 (μ_2)	13%
Standard deviation of equity 2 (σ_2)	30%
Correlation coefficient(ρ)	0
Bquest horizon (H)	20
Retirement wealth ratio (y)	0.4
Interest rate (r)	6%
Inflation rate (i)	3.5%
Ordinary income tax rate (τ_d)	36%
Capital gains tax rate (τ_g)	20%
Risk aversion parameter (γ)	3%
Fraction of total wealth earned as labor income (l)	15%
Fraction of labor income contributed to the TDRA (α)	20%

Table 2: Frequency and mean of various asset holdings when retirement wealth ratio is low ($y = .1$)

Magnitude (%)	Equity in the TBA	Equity in the TDRA	Bond in the TBA	Bond in the TDRA
$> 10\%$	100	0.00	0.00	100
$> 20\%$	100	0.00	0.00	100
$> 30\%$	100	0.00	0.00	100
$> 40\%$	100	0.00	0.00	100
$> 50\%$	100	0.00	0.00	100
$> 60\%$	100	0.00	0.00	100
$> 70\%$	100	0.00	0.00	100
Mean holding	105%	0%	-14%	100%

The frequency of various asset holdings, and their magnitude in different accounts over lifetime are presented in the table. This table is based on the values of the base case parameters. The investor is assumed to have low retirement wealth ratio (y), and she does not face any borrowing constraint. The first column indicates the magnitude of the wealth in a particular account allocated to a particular asset class. And the rest of the columns contain the frequencies of holdings of assets of particular asset class over lifetime in different accounts. The bottom row contains the mean holdings in particular asset class.

Table 3: Frequency and mean of various asset holdings when retirement wealth ratio is high ($y = .8$)

Magnitude (%)	Equity in the TBA	Equity in the TDRA	Bond in the TBA	Bond in the TDRA
$> 10\%$	100	100	0.00	100
$> 20\%$	100	100	0.00	100
$> 30\%$	100	100	0.00	100
$> 40\%$	100	100	0.00	100
$> 50\%$	100	86.25	0.00	13.75
$> 60\%$	100	0.00	0.00	0.00
$> 70\%$	100	0.00	0.00	0.00
Mean holding	170%	54%	-100%	46%

The frequency of various asset holdings, and their magnitude in different accounts over lifetime are presented in the table. This table is based on the values of the base case parameters. The investor is assumed to have high level of retirement wealth, and she does not face any borrowing constraint. The first column indicates the magnitude of the wealth in a particular account allocated to a particular asset class. And the rest of the columns contain the frequencies of holdings of assets of particular asset class over lifetime in different accounts. The bottom row contains the mean holdings in particular asset class.

Table 4: Change in utility due to the variation in retirement contribution level

y	$\Delta v_{5-20\%}$ with borrowing constraint	$\Delta v_{5-20\%}$ without borrowing constraint	$\Delta v_{20-30\%}$ with borrowing constraint	$\Delta v_{20-30\%}$ without borrowing constraint
.1	28.06%	52.80%	17.09%	35.19%
.4	17.88%	50.67%	11.92%	33.76%
.7	17.63%	17.17%	11.75%	11.44%

This table presents the average percentage changes in the utility level (during the working years) due to the changes in the retirement contribution limit. The changes are measured by measuring the changes in value of the the value function due to the changes in the retirement contribution level parameter, α. The retirement wealth ratios, y, are listed in the first column. The second column lists changes in the utility due to the change in the contribution level from 5% to 20% when borrowing is not allowed, and the third column lists changes in the utility due to the change in the contribution level from 5% to 20% when borrowing is allowed. In this case, the investors are allowed to borrow 100% of the amount of wealth they have in their brokerage account. Similarly, columns four and five list the change in utility due to the change in the contribution level from 20% to 30%, with and without the borrowing constraint, respectively.

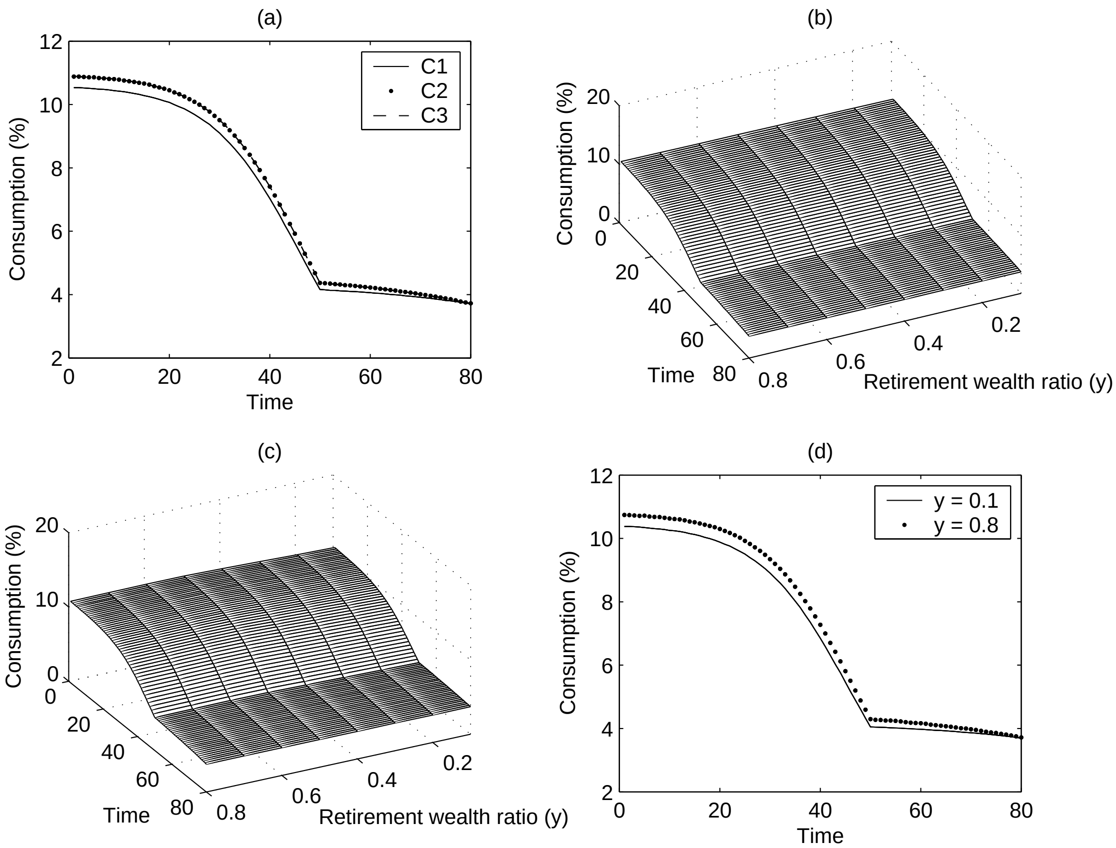

Figure 1: Parameter values are of the base portfolio. (a) Consumption under various restrictions over time. C1 represents consumption under both short sale and borrowing constraints, C2 represents consumption with only short sale constraint, and C3 represents consumption without borrowing or short sale constraints. (b) Allocation for consumption with borrowing and short sale constraints across time and retirement wealth ratio under borrowing constraint. (c) Allocation for consumption with borrowing and short sale constraints across time and retirement wealth ratio under no borrowing constraint. (d) Allocation for consumption for individuals with low and high retirement wealth ratio facing borrowing constraint.

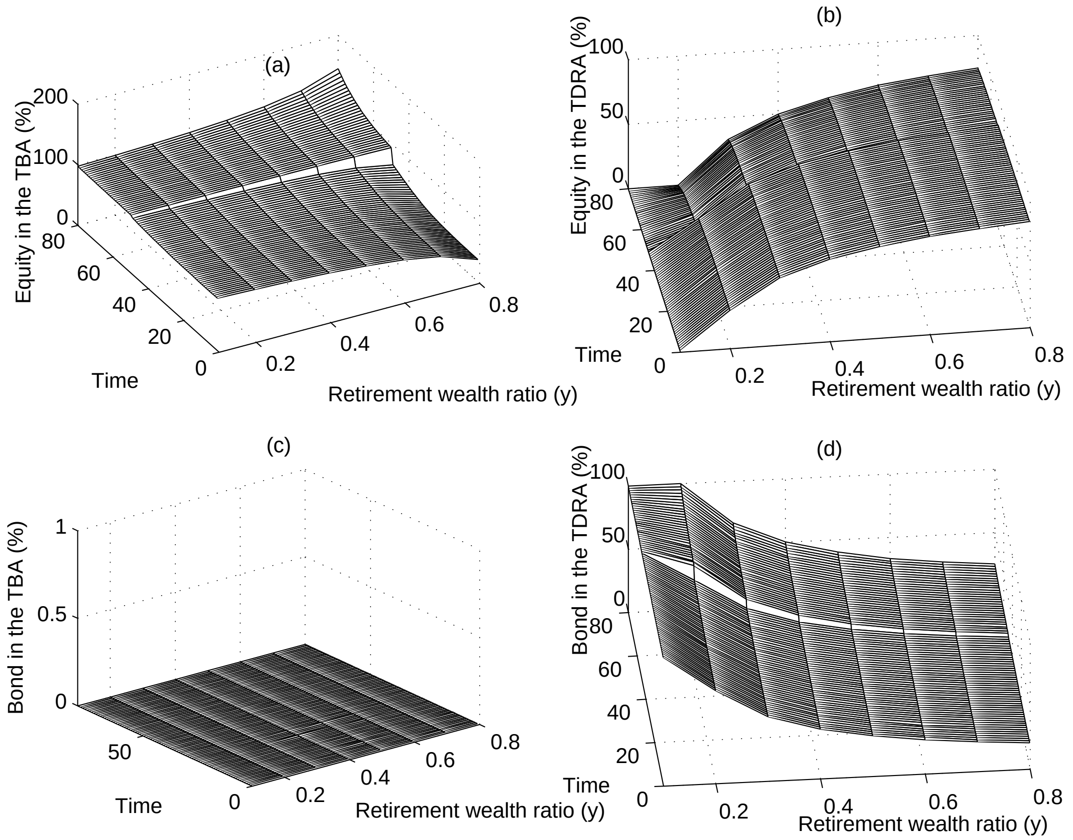

Figure 2: Parameter values are of the base portfolio with borrowing and short sale constraints. (a) Combined equity holding in the TBA across time and retirement wealth ratio. (b) Combined equity holding in the TDRA across time and retirement wealth ratio. (c) Bond holding in the TBA across time and retirement wealth ratio. (d) Bond holding in the TDRA across time and retirement wealth ratio.

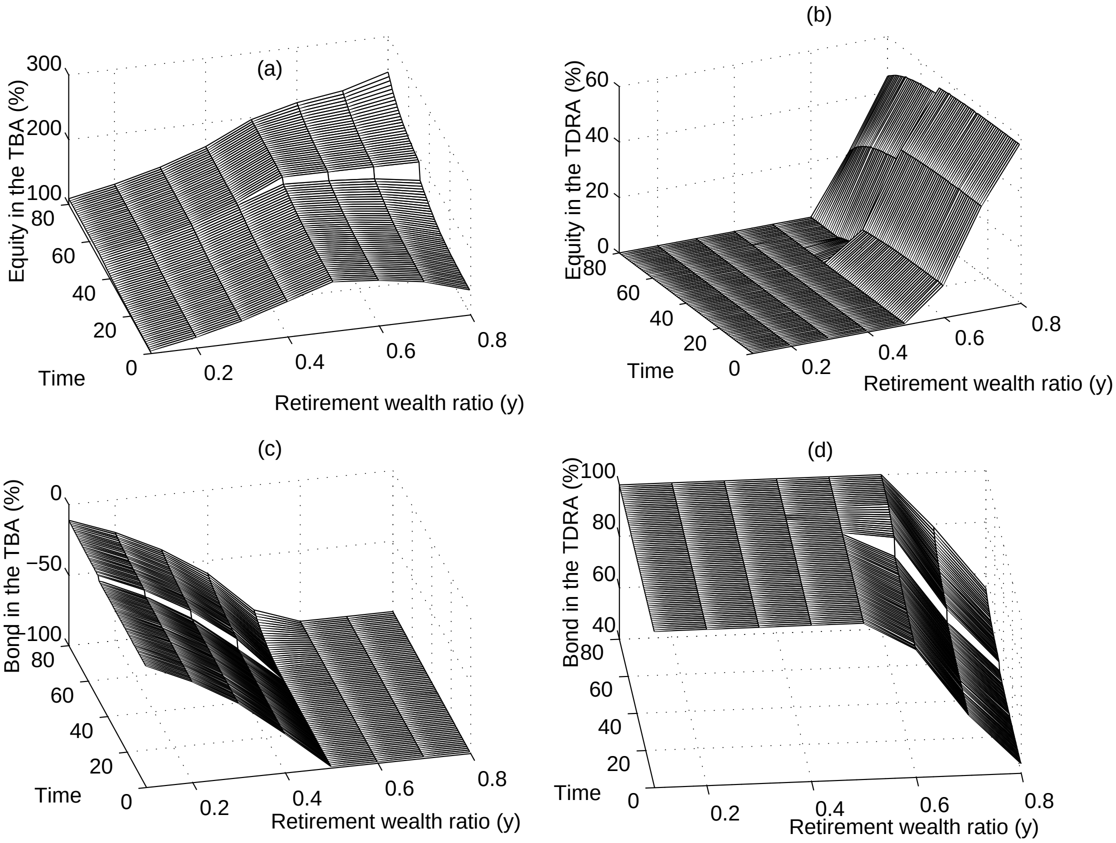

Figure 3: Parameter values are of the base portfolio without any borrowing constraint.(a) Combined equity holding in the TBA across time and retirement wealth ratio. (b) Combined equity holding in the TDRA across time and retirement wealth ratio. (c) Bond holding in the TBA across time and retirement wealth ratio. (d) Bond holding in the TDRA across time and retirement wealth ratio.

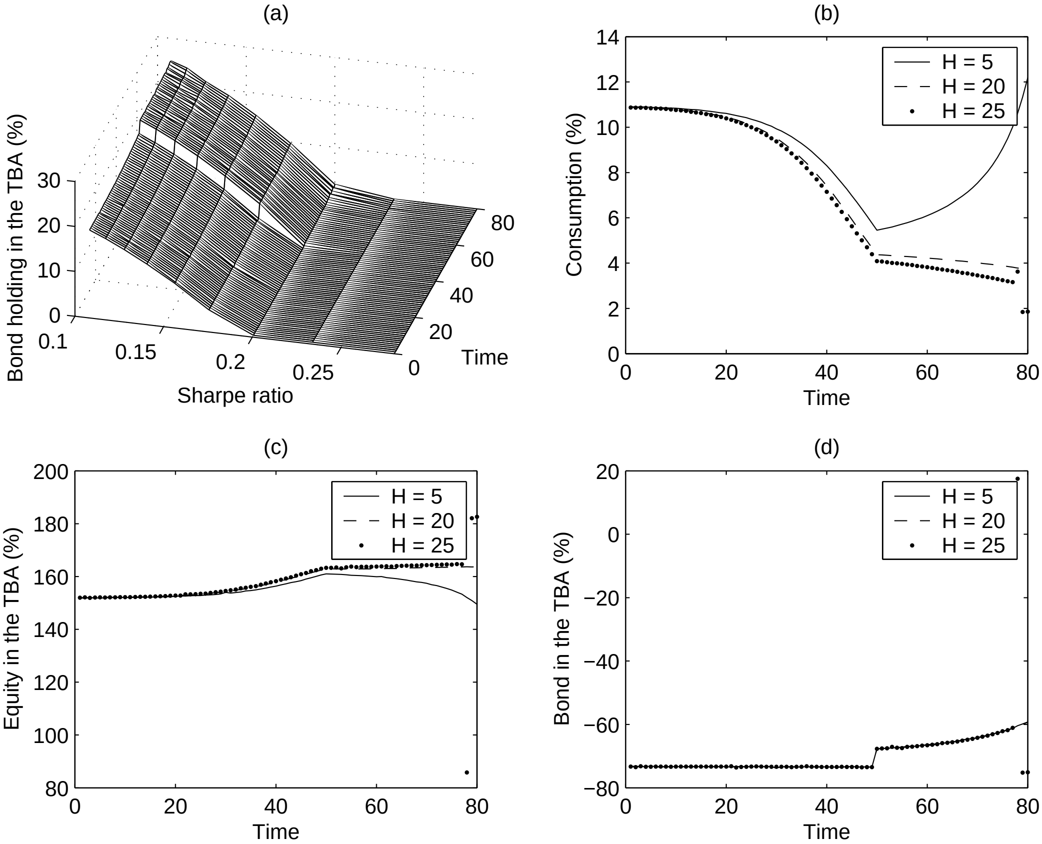

Figure 4: Parameter values are of the base portfolio. (a) Relationship between bond holding in the TBA and Sharpe ratio of the second risky asset. A vector of standard deviations, $\sigma_2 \in \{25\%, 30\%, 35\%, 40\%, 45\%, 50\%, 55\%, 60\%, 65\%\}$, is used to generate the Sharpe ratios. Then the Sharpe ratios are plotted against the bond holding in the TBA when borrowing is not allowed. In panels (b), (c), and (d), we consider an investor with average retirement wealth ratio, $y = .4$. Panel (b) presents consumption allocation under various bequest motives. Panel (c) presents combined equity holding in the TBA for various bequest motives. Panel (d) presents bond holding in the TBA for various bequest motives.

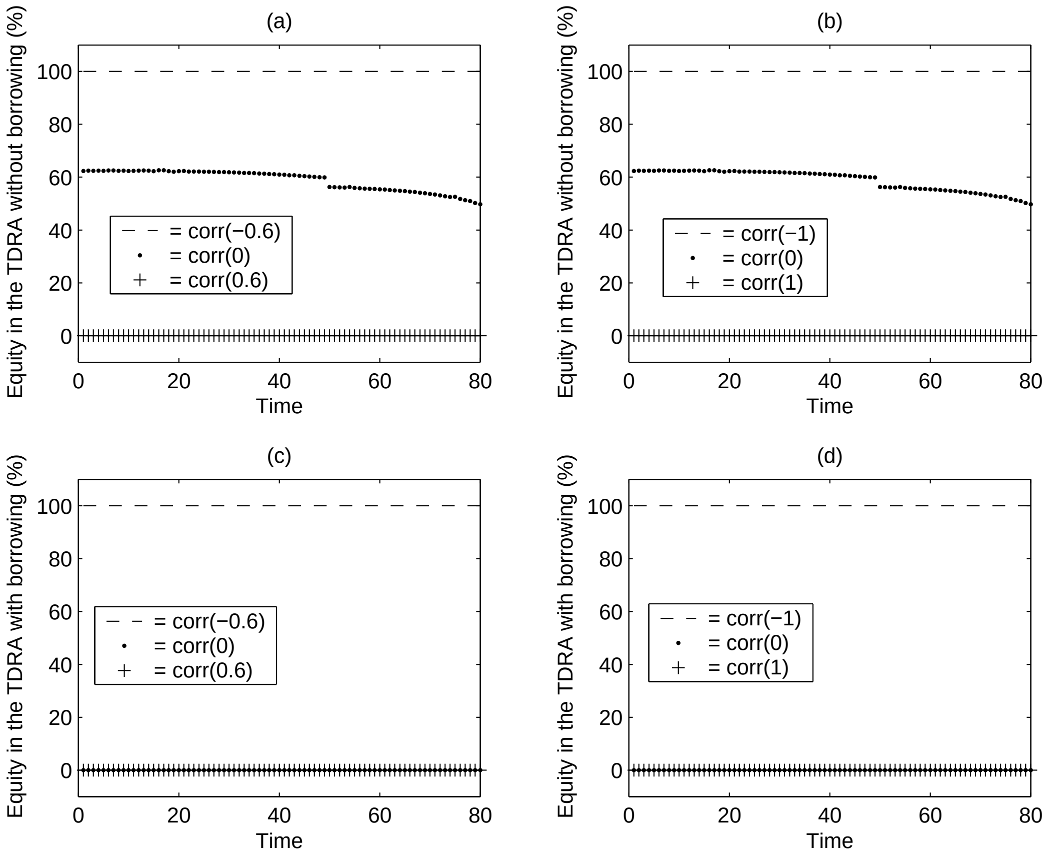

Figure 5: Parameter values are of the base portfolio. The plots present the impact of correlation structure of the equities on the level of equity holding in the TDRA. Panel (a) and (b) presents the case when borrowing is not allowed, and panel (c) and (d) present the case when the investor is allowed to borrow 100% of the wealth in her brokerage account. Panel (a) presents three curves indicating the levels of equity holding in the TDRA depending on correlation coefficeints of -0.6, 0, and .6. Panel (b) presents three curves indicating the levels of equity holding in the TDRA depending on correlation of -1, 0, and 1. Panel (c) and (d) are interpreted similarly.

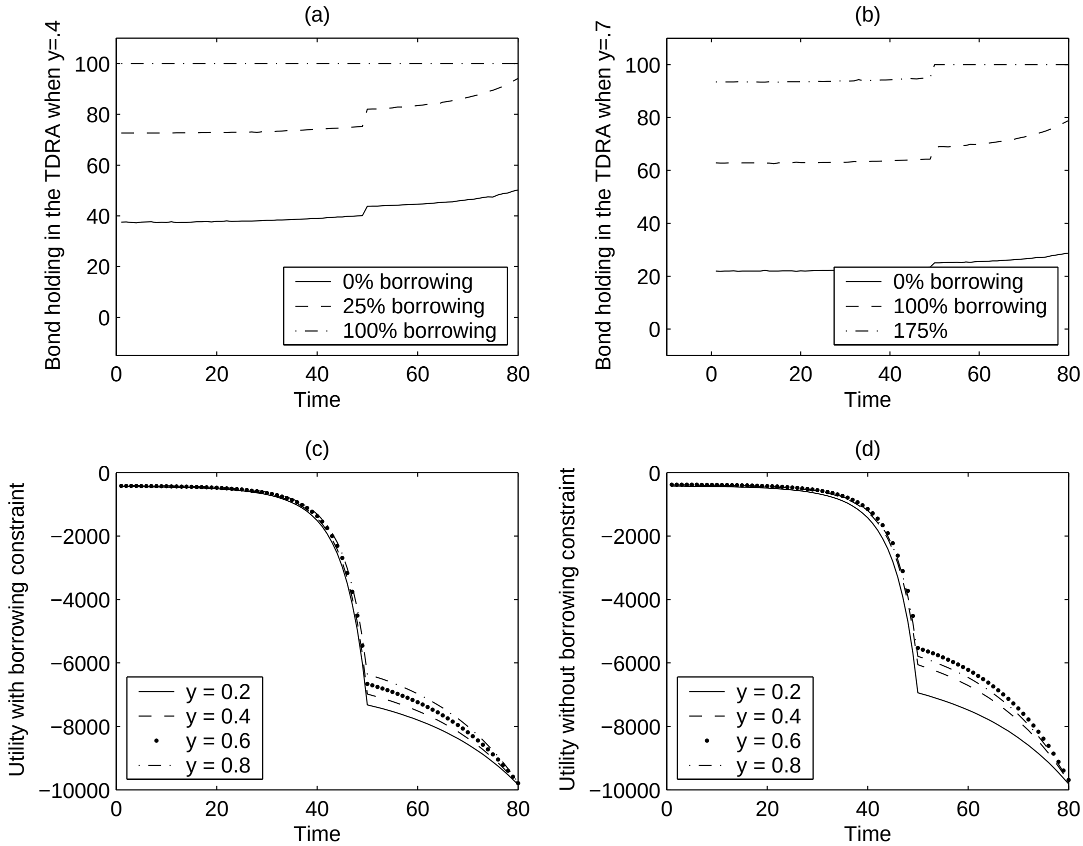

Figure 6: Parameter values are of the base portfolio. The plots in panel (a) and (b) depict the level of borrowing constraint relaxation, depending on the retirement wealth ratio (y), required to have all in bonds in the TDRA. Panel (a) presents the plot for $y = .4$. Three different curves in the plot show the levels of bond holding in the TDRA depending on the borrowing constraint relaxation. Three curves represnt the cases when no borrowing is allowed (0% borrowing), when the investor is allowed to borrow 25% of the wealth in her brokerage account, and when the investor is allowed to borrow 100% of the wealth in her brokerage account. Panel (b) is interprated in similar manner. Panel (c) and (d) present the utility levels due to variations in the retirement wealth ratios, y, with and without borrowing constraints. This is the presentation of our limited analysis of the optimal size of the retirement account. The plots indicate the optimal size of the retirement account depending on the borrowing constraints