FINE6860: Lecture #4
Models of Risky Investments

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1 Recent Stock Market History

Nominal Investment Returns over Ten Years:			
YEAR	Stocks (SP500)	Cash (T.Bills)	Inflation (CPI)
1995	37.43%	5.60%	2.54%
1996	23.07%	5.21%	3.32%
1997	33.36%	5.26%	1.92%
1998	28.58%	4.86%	1.61%
1999	21.04%	4.68%	2.68%
2000	-9.11%	5.89%	3.39%
2001	-11.88%	3.83%	1.55%
2002	-22.10%	1.65%	2.38%
2003	28.70%	1.02%	1.88%
2004	10.00%	2.00%	1.50%
Table #1 Source: Ibbotson Associates			

Invested in:	$1 Invested in the SP500 Index...Grows to:				
Jan 1995	$1.000				
Jan 1996	$1.374				
Jan 1997	$1.691	$1.000			
Jan 1998	$2.256	$1.334			
Jan 1999	$2.900	$1.715	$1.000		
Jan 2000	$3.510	$2.076	$1.210		
Jan 2001	$3.191	$1.886	$1.100	$1.000	
Jan 2002	$2.812	$1.662	$0.969	$0.881	
Jan 2003	$2.190	$1.295	$0.755	$0.686	$1.000
Jan 2004	$2.819	$1.667	$0.972	$0.883	$1.287
Jan 2005	$3.101	$1.833	$1.069	$0.972	$1.416
Growth (cc):	**11.32%**	**7.58%**	**1.11%**	**-0.71%**	**17.38%**
Table #2: Source: IFID Centre Calculations					

Table #1 can be converted from "nominal" pre-inflation numbers to "real" after-inflation numbers by dividing one-plus the investment return into one-plus the inflation rate and then subtracting one, via the formula $(1 + R)/(1 + \pi) - 1$. The intuition for the division, as opposed to subtracting inflation from the return, is the same logic for compounding interest. Table #3 displays the converted numbers.

After-inflation (Real) Returns over Ten Years:		
YEAR	Stocks (SP500)	Cash (T.Bills)
1995	34.03%	2.98%
1996	19.12%	1.83%
1997	30.85%	3.28%
1998	26.54%	3.20%
1999	17.88%	1.95%
2000	-12.09%	2.42%
2001	-13.23%	2.25%
2002	-23.91%	-0.71%
2003	26.33%	-0.84%
2004	8.37%	0.49%
Table #3 Source: IFID Centre calculations		

2 A Simple Experiment for Next Year?

Next Year?	Probability: (Q_i)	Return: (R_i)
Possibility #1	1/2	10%
Possibility #2	1/4	+35%
Possibility #3	1/4	-10%
Table #4: The Odds of a Simple Future		

What is the average or expected outcome for next year?

3 Arithmetic Average vs. Geometric Average

The arithmetic mean of these three numbers is:

$$\left(\frac{1}{2}\right)(+10\%) + \left(\frac{1}{4}\right)(+35\%) + \left(\frac{1}{4}\right)(-15\%) = 10.0\%. \quad (1)$$

What does this number actually mean? Explain...

Mathematically, the expected value of the investment return is the product

$$E[(1+R_1)(1+R_2)(1+R_3)..(1+R_n)] = E^n[(1+R)] \quad (2)$$

In contrast, the geometric mean is:

$$(1+0.10)^{(1/2)}(1+0.35)^{(1/4)}(1-0.15)^{(1/4)} - 1 = 8.55\% \quad (3)$$

Table #5 displays the geometric mean of a number of related "gamble" or investment opportunities:θ

What is the Geometric Mean Return?			
Prob. = 1/2	Prob. = 1/4	Prob. = 1/4	**G.M.**
10%	42%	-15%	10.0%
10%	35%	-15%	8.55%
10%	40%	-20%	7.89%
10%	45%	-25%	7.10%
10%	50%	-30%	6.17%
10%	70%	-51%	0.00%
Table #5:			

4 A Long-Term Model for Risk

I am now ready to present "the model" we will be using to describe the long-term evolution of indices or investment portfolios for (most of) the remainder of the analysis. We start with an initial investment of $S_0 = 100$, for example, and after T-years this capital amount grows to a random value:

$$S_T = S_0 e^{\widetilde{g}T}, \quad (4)$$

where the symbol $\widetilde{g}$ denotes the annualized *growth rate* of the portfolio during the T-year period. Stated differently, the time-scaled log-price-ratio is: $\ln[S_t/S_0]/T = \widetilde{g}$. Thus, for example, after $T = 10$ years the random growth rate might be a realized 8.5%, but after $T = 20$ years it is only 7.0%. In this case the portfolio or index might grow from $S_0 = 100$ to $S_{10} = 100e^{(0.085)(10)} = \$233.96$ after ten years, and to $S_{20} = 100e^{(0.07)(20)} = 405.52$ after twenty years.

I will assume that $\widetilde{g}$ is normally distributed with an expected value of ν (Greek letter nu) and a variance of σ^2/T, or standard deviation of $\sigma/\sqrt{T}$, where T is the relevant horizon over which we are forecasting investment returns.

More formally, I will assume that

$$\widetilde{g} \sim \mathbf{N}(\nu, \sigma^2/T), \tag{5}$$

and therefore $\widetilde{g}T \sim \mathbf{N}(vT, \sigma^2 T)$. Figure #1 provides a graphical illustration of the probability density function (PDF) curves for various values of T. Note that as T increases, the dispersion around the 7% declines in proportion to $1/\sqrt{T}$.

In the language of Microsoft Excel, we are using the function `NORMSDIST(0,`$\nu, \sigma/\sqrt{T}$`,TRUE)`, with different values of ν, σ and T.

		The Probability of Losing Money:				
ν	σ	T=1	T=5	T=10	T=20	T=30
12%	20%	0.274	0.090	0.029	0.004	0.001
12%	10%	0.115	0.004	0.000	0.000	0.000
7%	20%	0.363	0.217	0.134	0.059	0.028
7%	10%	0.242	0.059	0.013	0.001	0.000
5%	20%	0.401	0.288	0.215	0.132	0.085
5%	10%	0.309	0.132	0.057	0.013	0.003

Table #6: Source IFID Centre Calculations

5 Introducing Brownian Motion

The quantity $\widetilde{g}t$ is extremely important in it's own right. The product of the (random) growth rate and time – which is $\ln[S_t/S_0]$ using our first definition – often has it's own notation and description amongst financial specialists. We will adopt the convention and define a new expression:

$$\mathbf{B}_t^{(\nu,\sigma)} := \widetilde{g}t \sim \mathbf{N}(\nu t, \sigma^2 t), \tag{6}$$

which we will abbreviate by $\mathbf{B}_t$ when $\nu = 0$ and $\sigma = 1$, instead of using $\mathbf{B}_t^{(0,1)}$. The object $\mathbf{B}_t^{(\nu,\sigma)}$ is normally distributed with an expected value of νt and a standard deviation of $\sigma\sqrt{t}$.

In fact, the standard Brownian motion process $\mathbf{B}_t$ can be used to construct the more complex $\mathbf{B}_t^{(\nu,\sigma)}$ via the linear relationship defined by:

$$\mathbf{B}_t^{(\nu,\sigma)} = \sigma\mathbf{B}_t + \nu t. \tag{7}$$

At first it might seem odd, but think about this for a while.

There is an important theoretical property of B_t that has some investment implications and is therefore worth discussing:

$$\lim_{t\to\infty} \frac{\mathbf{B}_t}{t} \to 0. \tag{8}$$

Along the same lines, if you remember the construction of the non-standard Brownian motion $\mathbf{B}_t^{(\nu,\sigma)}$, which is constructed from the standard Brownian motion $\mathbf{B}_t$ scaled by σ and then adding νt, we have:

$$\lim_{t\to\infty} \widetilde{g} = \frac{\mathbf{B}_t^{(\nu,\sigma)}}{t} = \nu + \sigma \frac{\mathbf{B}_t}{t} \to \nu. \tag{9}$$

The intuition is the same.

6 Index Averages and Index Medians:

At this point you should have a decent idea of how the fundamental object $\mathbf{B}_t$ behaves over time. In this section we delve into the behavior of e^{B_t}, which represents the evolution of the index (or portfolio) value itself. Remember the various stages in our definition:

$$S_t = S_0 e^{\widetilde{g}t} := S_0 e^{\mathbf{B}_t^{(\nu,\sigma)}} = S_0 e^{\nu t + \sigma \mathbf{B}_t}. \quad (10)$$

The last two equalities come from the construction of the Brownian motion. When $\sigma = 0$ the index or portfolio will grow at a fixed rate of ν , with zero uncertainty or randomness.

I am now interested in some of the probabilistic properties of S_T itself. The median value of the index at time t is the simple and intuitive:

$$M[S_t] = S_0 M[e^{\nu t + \sigma \mathbf{B}_t}] = S_0 e^{\nu t}.$$

Thus, 50% of the time you will see S_t above $S_0 e^{\nu t}$ and 50% of time S_t will fall below $S_0 e^{vt}$. As time $t \to \infty$, the median value of the index or portfolio grows without bound provided that $\nu > 0$. If $\nu = 0$, the median value of $S_t = 0$ for all values of t, since there is no growth.

You can verify that indeed the median value for S_t is $S_0 e^{\nu t}$, by going thru the following steps. First, note that the general probability:

$$\begin{aligned} \Pr[S_t \ \leq \ u] &= \Pr[\ln[S_0] + vt + \sigma \mathbf{B}_t \leq \ln[u]] \\ &= \ \Pr[\frac{\mathbf{B}_t}{\sqrt{t}} \leq \frac{\ln[u/S_0] - vt}{\sigma \sqrt{t}}]. \end{aligned} \tag{11}$$

By construction and definition of the standard Brownian motion, the term $\mathbf{B}_t/\sqrt{t}$ is Normally distributed with an expected value of zero and a standard deviation of one. This leads to:

$$\Pr[S_t \leq u] = \int_{-\infty}^{\frac{\ln[u/S_0]-vt}{\sigma\sqrt{t}}} \frac{e^{-\frac{1}{2}(z)^2}}{\sqrt{2\pi}} dz, \tag{12}$$

where the integrand should be recognized as the basic Gaussian probability density function. Thus, if we make the substitution: $u = S_0 e^{\nu t}$, the upper bound of integration collapses to zero, which by symmetry of the Normal distribution around zero, leads to an integral value of $\Pr[S_t < S_0 e^{\nu t}] = 1/2$, and hence it is the median value.

In contrast, the expected value $E[S_t]$ must be computed by integrating:

$$\begin{aligned} E[S_t] &= \int_{-\infty}^{+\infty} S_0 \left(e^{\nu t + \frac{1}{2}\sigma \sqrt{t} z} \right) \frac{e^{-\frac{1}{2}(z)^2}}{\sqrt{2\pi}} dz \\ &= S_0 e^{(\nu + \frac{1}{2}\sigma^2)t}. \end{aligned} \tag{13}$$

The first portion of the integrand contains the exponentiation which is then multiplied by the normal density.

7 Probability of Regret

Based on the same logic we used earlier to compute median and mean values, note that:

$$\begin{aligned} \Pr[S_t \ \le \ S_0 e^{rt}] &= \Pr[\frac{\mathbf{B}_t}{\sqrt{t}} \le -\left(\frac{\nu - r}{\sigma}\right)\sqrt{t}] \\ &= \varphi\left(-\left(\frac{\nu - r}{\sigma}\right)\sqrt{t}\right), \end{aligned} \tag{14}$$

where $\varphi(z)$ denotes the CDF of the standard normal.

8 Rate of Change

We are now in a position to investigate the "rate of change" of the index or portfolio over time.

$$\frac{\Delta S_i}{S_i} = (\nu + \frac{1}{2}\sigma^2)\Delta t + \sigma \Delta B_i \tag{15}$$

The expression $(\nu + \frac{1}{2}\sigma^2)$ is central to a number of formulas in finance, which is why it common to see this expression defined as:

$$\mu = \nu + \frac{1}{2}\sigma^2 \quad \Longleftrightarrow \quad \nu = \mu - \frac{1}{2}\sigma^2 \tag{16}$$

The parameter μ is often labeled the (c.c.) arithmetic mean and the ν is labelled the (c.c.) geometric mean. Remember that the arithmetic mean is larger than the geometric mean by a factor of $\frac{1}{2}\sigma^2$.

Either way, this enables us to simulate portfolio values. Explain.

In differential notation the dynamic evolution of the portfolio is described by the Stochastic Differential Equation (SDE):

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{17}$$

I will move between the two notations, using μ and ν depending on the need and context.

9 Asset Allocation

I start with a collection of n securities and let S_t^i denote the price of the i'th security at time t. The evolution of each individual S_t^i is modeled by the Stochastic Differential Equation (SDE) introduced above, which can be represented by:

$$dS_t^i = \mu_i S_t^i dt + \sigma_i S_t^i d\mathbf{B}_t^i, \tag{18}$$

where $\mathbf{B}_t^i$ is now a vector of standard Brownian motions, and without any loss of generality, I scale $S_0^i = 1$ for all securities $i \leq n$. The parameters $\{\mu_i, \sigma_i\}$ denote the instantaneous drift rate (a.k.a. mean) and diffusion coefficient (a.k.a. volatility) of the i'th security. The correlation coefficient is denoted by $d\left\langle \mathbf{B}^i, \mathbf{B}^j \right\rangle = \rho_{ij} dt$, with the understanding that $\rho_{ij} = \rho_{ji}$, and $\rho_{ii} = \rho_{jj} = 1$ for all $i, j \leq n$.

Recall that equation (18) can also be written as:

$$S_t^i = e^{\left(\mu_i - \sigma_i^2/2\right)t + \sigma_i \mathbf{B}_t^i} = e^{(\nu_i)t + \sigma_i \mathbf{B}_t^i}, \tag{19}$$

with expectation $E[S_t^i | S_0^i = 1] = e^{\mu_i t}$ and with standard deviation $SD[S_t^i | S_0^i = 1] = e^{\mu_i t}\sqrt{e^{\sigma_i^2 t} - 1}$. Once again, the log-price is normally distributed with mean $E[\ln[S_t^i]] | S_0^i = 1] = (\mu_i - \sigma_i^2/2)t$, and standard deviation $SD[\ln[S_t^i]] | S_0^i = 1] = \sigma_i \sqrt{t}$.

By simple construction, the portfolio process W_t will obey a Stochastic Differential Equation (SDE) denoted by:

$$
\begin{aligned}
dW_t &= \sum_{i=1}^{n} \alpha_i W_t \left(\frac{dS_t^i}{S_t^i} \right) \\
&= \sum_{i=1}^{n} \alpha_i \mu_i W_t dt + \sum_{i=1}^{n} \alpha_i \sigma_i W_t d\mathbf{B}_t^i. \qquad (20)
\end{aligned}
$$

Under this representation, the aggregate portfolio process W_t is 'driven' by n correlated standard Brownian motion factors $\mathbf{B}_t^i$, where $i = 1...n$. However, equation (20) can be simplified by combining the n distinct factors into one independent source of risk.

To this end, we can define a new portfolio drift coefficient:

$$\mu_p(n) = \sum_{i=1}^{n} \alpha_i \mu_i. \tag{21}$$

Also, we can simplify the Brownian components in equation (20) by defining an aggregate portfolio standard deviation of volatility via:

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i \sigma_i d\mathbf{B}_t^i &= \left[\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \sigma_i \rho_{ij} \sigma_j \alpha_j} \right] d\mathbf{B}_t \\
&= \left[\sqrt{\sum_{k=1}^{n} \alpha_k^2 \sigma_k^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \alpha_i \sigma_i \rho_{ij} \sigma_j \alpha_j} \right] d\mathbf{B}_t \\
&= \sigma_p(n) d\mathbf{B}_t.
\end{aligned} \tag{22}$$

The new combined (source of risk) term $d\mathbf{B}_t$ is a standard one-dimensional Brownian motion. The new $\sigma_p(n)$ is the portfolio volatility which is an explicit function of the size (space dimension) n of the portfolio, and an implicit function of the volatility and correlation structure, as well as the individual security weights.

The resulting stochastic differential equation (SDE) obeyed by the (total wealth) portfolio can be represented by:

$$dW_t = \mu_p(n) W_t dt + \sigma_p(n) W_t d\mathbf{B}_t, \qquad W_0 = 1 \tag{23}$$

Akin to the case for individual securities, the explicit solution to the stochastic differential equation in equation (23) is:

$$W_t = e^{\left(\mu_p(n) - \sigma_p^2(n)/2\right)t + \sigma_p(n)\mathbf{B}_t}, \tag{24}$$

where we now use the definition:

$$\nu_p(n) := \mu_p(n) - \frac{1}{2}\sigma_p^2(n) \tag{25}$$

10 Space-Time Diversification

We can now put two ideas together. If a portfolio – consisting of n securities – is held for a period of t years, the Probability of Regret is defined equal to:

$$\mathbf{PoR}(n,t) \ := \Pr[W_t \leq e^{rt}] \ = \Pr\left[\ln\left[W_t\right] \leq rt\right], \tag{26}$$

which is the probability of doing *worse* than the interest rate r. By the definition of W_t from equation (24), we arrive at:

$$\begin{aligned} \mathbf{PoR}(n,t) \ &= \ \Pr\left[\frac{B_t}{\sqrt{t}} \leq -\left(\frac{\nu_p(n) - r}{\sigma_p(n)}\right)\sqrt{t}\right] \\ &= \ \varphi\left(-\left(\frac{\nu_p(n) - r}{\sigma_p(n)}\right)\sqrt{t}\right), \end{aligned} \tag{27}$$

which is identical to the earlier expression.

To obtain more precise results, let us now assume that $\alpha_i = 1/n$, which means that the initial wealth $W_0 = w$ is portioned and invested equally amongst the n securities, and maintained in those proportions during the entire time $[0,t]$. Furthermore, assume that all securities in the portfolio have the same drift rate μ, the same volatility σ, and a uniform correlation structure denoted by ρ. In other words, the covariance matrix $\mathbf{\Sigma}$, for the n securities can be represented as:

$$\mathbf{\Sigma} := \begin{pmatrix} \sigma^2 & & & . & \rho\sigma^2 \\ & \sigma^2 & & . & \rho\sigma^2 \\ & & \sigma^2 & . & \rho\sigma^2 \\ . & . & . & . & . \\ \rho\sigma^2 & \rho\sigma^2 & \rho\sigma^2 & . & \sigma^2 \end{pmatrix} \tag{28}$$

This structure may seem odd at first. But, my objective is to examine the impact of adding more securities (space), and holding them for longer periods (time), on the $\mathbf{PoR}(n,t)$. In any event, according to equation (22), the portfolio variance, which we denote explicitly by $\sigma_p^2(n|\sigma,\rho)$, collapses to:

$$\begin{aligned}\sigma_p^2(n|\sigma,\rho) &= \sum_{k=1}^{n}\left(\frac{1}{n}\right)^2\sigma^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\left(\frac{1}{n}\right)^2\rho\sigma^2 \\ &= n\frac{\sigma^2}{n^2} + (n^2-n)\rho\frac{\sigma^2}{n^2} \\ &= \frac{\sigma^2}{n} + \left(1-\frac{1}{n}\right)\rho\sigma^2 = \sigma^2\left(\frac{1}{n}(1-\rho)+\rho\right) \qquad (29)\end{aligned}$$

Therefore, the portfolio volatility, which is the diffusion coefficient of the wealth process W_t, is:

$$\sigma_p(n|\sigma,\rho) = \sigma\sqrt{\rho + \frac{1-\rho}{n}} \qquad (30)$$

As one expects intuitively, the derivative of the portfolio volatility $\sigma_p(n|\sigma,\rho)$, with respect to the space variable n, is:

$$\begin{aligned}\frac{\partial\sigma_p(n|\sigma,\rho)}{\partial n} &= \frac{\sigma^2(\rho-1)}{2n^2\sqrt{\rho+(1-\rho)/n}} \\ &= \frac{\sigma^2(\rho-1)}{2n^2\sigma_p(n|\sigma,\rho)} < 0, \qquad \forall\rho<1, \qquad (31)\end{aligned}$$

which implies the obvious conclusion that a larger portfolio reduces volatility.

Along the same lines, we have that:

$$\begin{aligned}\frac{\partial \sigma_p(n|\sigma,\rho)}{\partial \rho} &= \frac{\sigma^2(n-1)}{2n\sqrt{\rho + (1-\rho)/n}} \\ &= \frac{\sigma^2(n-1)}{2n\sigma_p(n|\sigma,\rho)} > 0, \qquad \forall n > 1 \qquad (32)\end{aligned}$$

which implies that a larger correlation coefficient, *ceteris paribus*, leads to a larger portfolio volatility and a corresponding increase in the shortfall **PoR**(n,t). Finally, it should be obvious from equation (30) that the derivative of $\sigma_p(n|\sigma,\rho)$ with respect to σ, is positive as well.

Note that as a result of the square root in equation (30), we are forced into a condition that:

$$\rho + \frac{1-\rho}{n} \geq 0 \quad \Longrightarrow \quad \rho \geq 1/(1-n). \qquad (33)$$

Finally, the probability of regret, as per equation (**??**), will be:

$$\mathbf{PoR}(n, t|r, \mu, \sigma, \rho) = \varphi\left(\frac{r - \mu + \sigma^2\left(\rho + (1-\rho)/n\right)/2}{\sigma\sqrt{\rho + (1-\rho)/n}}\sqrt{t}\right), \tag{34}$$

where the explicit variables r, μ, σ, ρ are introduced to denote the homogenous case of constant parameters and equal portfolio weights.