FINE6860: Lecture #6
Models for Life Insurance

By: Moshe A. Milevsky

Summer 2006

1 Life Insurance Prices

U.S. Monthly Premiums for $100,000 of Life Insurance

Term of	Age 30		Age 50		Age 70	
Insurance	*Male*	*Fem.*	*Male*	*Fem.*	*Male*	*Fem.*
5 years	12.71	11.53	19.65	15.30	105.65	59.27
10 years	8.21	7.68	17.95	14.57	102.51	55.96
20 years	11.01	9.68	27.56	21.19	207.54	128.07
30 years	15.47	12.88	46.23	33.15	307.33	259.50
Term-to-100*	33.51	27.27	103.60	81.51	373.83	299.07

Table #1: Source: Compulife, "Preferred Health" applicant

*Canadian data, "Regular Health," non-smoker

2 The Impact of Health Status

U.S. Monthly Premiums for $100,000 of Life Insurance								
50 year-old with varying Health Status								
Inc.	Average		Above Avg.		Excellent		Exceptional	
Term	*Male*	*Fem.*	*Male*	*Fem.*	*Male*	*Fem.*	*Male*	*Fem.*
5 yrs	27.61	20.68	25.16	17.49	19.65	15.30	15.37	12.11
10 yrs	23.54	18.38	22.64	17.94	17.95	14.57	14.86	12.48
20 yrs	38.69	28.65	35.30	26.73	27.56	21.19	23.85	17.90
Table #2: Source: Compulife (webstite www.term4sale.com)								
Note: Non-Smoking,								

3 How Much Life Insurance Do You Need?

What is the value of your human capital?

4 Value of Life Insurance: Net Single Premium

Conceptually, here is the main idea behind the pricing of life insurance. If the valuation rate is $r = 5\%$ for example and the insured person dies at time $T = 10$ years, then the discounted value of the death benefit at time zero is: $e^{-(0.05)10} = \$0.606$ per dollar of face value. Stated differently, an initial premium of 0.606 dollars invested at a rate of $r = 5\%$ would grow to \$1 at the time of death, which would be enough to pay the death benefit to the beneficiary. If, on the other hand, the insured person dies at time $T = 20$ years, then the discounted value of the death benefit is a much lower $e^{-(0.05)20} = \$0.368$ per dollar of face value. In this case, an initial premium of 0.368 dollars is sufficient.

When the remaining lifetime random variable is $\mathbf{T}_x$, the stochastic discounted value (SDV) of a \$1 death benefit at a valuation rate r, is:

$$\mathbf{A}_x = e^{-r\mathbf{T}_x}. \tag{1}$$

The stochastic discounted value $\mathbf{A}_x$ is the life insurance counterpart to the stochastic discounted value $\mathbf{a}_x$ for the pension annuity. Recall that $\mathbf{a}_x$ was defined by the integral relationship:

$$\mathbf{a}_x = \int_0^{\mathbf{T}_x} e^{-rt} dt. \tag{2}$$

Parallel to the pension annuity factor which was defined by $\overline{a}_x = E[\mathbf{a}_x]$, we define the net-single premium (NSP) via:

$$\overline{A}_x = E\left[e^{-r\mathbf{T}_x}\right] = \int_0^\infty e^{-rt} f_x(t)dt, \tag{3}$$

where $f_x(t)$ denotes the probability density function (PDF) of the remaining lifetime random variable $\mathbf{T}_x$. The intuition is as follows. Starting from the perspective of the current age (x), we must add-up all possible discounted values e^{-rt}, weighted by the probability of death at that instant $f_x(t)$. The sum (integral) of these discounted values is the net-single premium. Stated differently, if you pay the fair actuarial premium $\overline{A}_x$ for life insurance coverage at age x, the discounted value of your expected profit is $E[\mathbf{A}_x - \overline{A}_x] = 0$.

5 Valuing Life Insurance using Pension Annuities

So, to compute A_x one must "do" some calculus again. The method of integration-by-parts which is at the heart of calculus leads to a very helpful shortcut for valuing the NSP for life insurance. Recall the basic relationship

$$\begin{aligned} \frac{d}{dt}(u(t)v(t)) &= u(t)dv(t) + v(t)du(t) \\ &\Longleftrightarrow \\ \int u(t)dv(t) &= u(t)v(t) - \int v(t)du(t), \end{aligned} \tag{4}$$

where both $u(t)$ and $v(t)$ are general functions of t, and $du(t), dv(t)$ denote derivatives with respect to t.

Using this insight, going back to equation (3), we can substitute $u(t) = e^{-rt}$ and $dv(t) = f_x(t)dt$ in the integrand. In this case, $du(t) = -re^{-rt}$, and $v(t) = F_x(t)$, based on the relationship between the CDF and PDF of remaining lifetime random variable. This leads us to the general relationship:

$$\int e^{-rt} f_x(t)dt = e^{-rt}F_x(t) - \int F_x(t)(-re^{-rt})dt. \tag{5}$$

The right-hand side of this equation can be written as:

$$= e^{-rt}F_x(t) - r\left(\int (1 - F_x(t))e^{-rt}dt - \int e^{-rt}dt\right), \tag{6}$$

by artificially adding and then subtracting an extra integral term. We then recognize $({}_tp_x) = (1 - F_x(t))$ in the integrand of the first integral as the conditional survival probability. In the end, this leaves us with:

$$\int e^{-rt} f_x(t)dt = e^{-rt}F_x(t) - r\int ({}_tp_x)e^{-rt}dt + 1, \tag{7}$$

which, when evaluated from the lower bound of $t = 0$ to the upper bound of $t = \infty$, leads to a very recognizable expression:

$$\overline{A}_x := \int_0^\infty e^{-rt} f_x(t)dt = 1 - r\overline{a}_x. \tag{8}$$

6 Arbitrage Relationship.

Individual borrows \$100 from a bank at an interest rate of r. This loan is structured as interest-only so that each year the borrower pays $100r$ in interest payments, which is $100rdt$. The loan principle is due and payable when the borrower dies. To cover this risk the borrower purchases a life insurance policy – with the bank as beneficiary – and pays $100\overline{A}_x$ for this coverage. The individual purchased a life annuity to cover the interest payments of $100rdt$. The left-overs after paying for life insurance and pension annuity was

$$100 - 100\overline{A}_x - 100r\overline{a}_x \tag{9}$$

Indeed, by dividing all terms by 100 and then isolating the NSP, this also implies the fundamental relationship between the net single premium and the annuity factor $\overline{A}_x = 1 - r\overline{a}_x$. In fact, another way of expressing this relationship is that:

$$\frac{\overline{A}_x}{\overline{a}_x} = \frac{1}{\overline{a}_x} - r \tag{10}$$

The ratio $\overline{A}_x/\overline{a}_x$ has its own special meaning and interpretation, one which we will return to later.

7 Value of Life Insurance: Exponential

For example, under exponential mortality where the IFM curve $\lambda(x) = \lambda$, the NSP becomes:

$$\overline{A}_x = 1 - \frac{r}{r+\lambda} = \frac{\lambda}{r+\lambda} \tag{11}$$

For example, when the life expectancy is $1/\lambda = 20$ and the valuation rate is $r = 5\%$, the NSP is equal to $\overline{A}_x = 0.05/(0.10) = 0.5$ per \$1 of life insurance protection. But, when the valuation rate $r = 10\%$, the NSP is $\overline{A}_x = 0.05/(0.15) = 0.333$ per \$1 of life insurance protection. As you would expect, increasing the valuation rate r tends to reduce the value of the NSP while increasing the IFM λ will increase the value of the NSP.

8 Value of Life Insurance: GoMa Mortality

Under the GoMa law of mortality, the value of $\overline{A}_x$ can be expressed as:

$$\overline{A}_x = 1 - \frac{rb\Gamma\left(-(\lambda + r)b, \exp\{\frac{x-m}{b}\}\right)}{\exp\left\{(m-x)(\lambda + r) - \exp\{\frac{x-m}{b}\}\right\}}, \tag{12}$$

where all I have done use used the relationship $\overline{A}_x = 1 - r\overline{a}_x$ and then "plugged in" the relevant pension annuity factor from the previous chapter.

For example, using our favorite $m = 86.34$, $b = 9.5, \lambda = 0$ GoMa parameters from the previous chapter the NSP under a $r = 5\%$ valuation rate is $\overline{A}_{30} = \$0.0962$ at age 30, $\overline{A}_{50} = \$0.239$ at age 50 and $\overline{A}_{70} = \$0.509$ at age 70. Each of these premiums will "buy" \$1 of life insurance protection. Quite obviously, at younger ages where $\lambda(x)$ is small, the life insurance cost is minimal, while at higher ages where $\lambda(x)$ is higher, the cost is higher as well. As a means of comparison, the pension annuity factor at the same ages and valuation rates would be $\overline{a}_{30} = 18.\,075$, $\overline{a}_{50} = 15.\,229$ and $\overline{a}_{70} = 9.\,822$ per dollar of lifetime annual income.

$100,000 of Life Insurance Protection			
How much Does it Cost (NSP) Up Front? $\overline{A}_x$			
Starting at:	$r = 4\%$	$r = 6\%$	$r = 8\%$
Age 35	$\$17,892$	$\$8460$	$\$4,376$
Age 45	$\$25,916$	$\$14,449$	$\$8,616$
Age 55	$\$36,711$	$\$23,800$	$\$16,161$
Age 65	$\$50,185$	$\$37,155$	$\$28,298$
Table #3: Source IFID Centre Calculations			
GoMa Mortality $m = 86.34$, $b = 9.5$			

9 Life insurance Paid by Installments

When the insurance is paid over time as opposed to all at once, the premium must be amortized or spread over the life of the insured. In the event of coverage that lasts a lifetime, the A_x must be converted into a pension annuity factor.

$$\text{NPP} := \frac{\overline{A}_x}{\bar{a}_x}. \tag{13}$$

Remember that $\frac{\overline{A}_x}{\bar{a}_x} = \frac{1}{\bar{a}_x} - r$ so that the NPP can be computed by taking the inverse of the pension annuity factor and subtracting off the valuation rate. In the case of exponential mortality this collapses to NPP = λ, which oddly enough does not depend on the valuation rate. It is purely a function of the instantaneous force of mortality. In the case of GoMa mortality, the NPP expression can again be computed quite easily.

<table>
<tr><td colspan="4">$100,000 of Life Insurance Protection</td></tr>
<tr><td colspan="4">How much Does it Cost (NPP) in Installments? $\frac{\overline{A}_x}{\overline{a}_x}$</td></tr>
<tr><td>Starting at:</td><td>$r = 4\%$</td><td>$r = 6\%$</td><td>$r = 8\%$</td></tr>
<tr><td>Age 35</td><td>$871.63</td><td>$554.51</td><td>$366.10</td></tr>
<tr><td>Age 45</td><td>$1,399.27</td><td>$1,013.32</td><td>$754.27</td></tr>
<tr><td>Age 55</td><td>$2,320.21</td><td>$1,874.00</td><td>$1,542.10</td></tr>
<tr><td>Age 65</td><td>$4,029.72</td><td>$3,547.26</td><td>$3,157.28</td></tr>
<tr><td colspan="4">Table #4: Source IFID Centre Calculations</td></tr>
<tr><td colspan="4">GoMa Mortality $m = 86.34$, $b = 9.5$</td></tr>
</table>

10 Permutations on a Theme

In some cases the life insurance is paid for now, but it only starts in τ years.

$$({}_u\overline{A}_x) := \int_u^\infty e^{-rt} f_x(t)dt \tag{14}$$

in other cases the insurance starts immediately, but is only valid for a pre-determined period of time:

$$\overline{A}_{x:\tau} := \int_0^\tau e^{-rt} f_x(t)dt \tag{15}$$

These definitions are parallel with

$$({}_u\overline{a}_x) := \int_u^\infty e^{-rt}(1 - F_x(t))dt \tag{16}$$

and

$$\overline{a}_{x:\tau} := \int_0^\tau e^{-rt}(1 - F_x(t))dt. \tag{17}$$

11 Duration and Convexity

Like in the case of pension annuities, we can compute the duration of the NSP and NPP using the following relationship:

$$\frac{\partial}{\partial r}\overline{A}_x = \frac{\partial}{\partial r}\left(1 - r\overline{a}_x\right) = -\left(r\frac{\partial}{\partial r}\overline{a}_x + \overline{a}_x\right) \tag{18}$$

The tradition is to define duration D as the "negative" of this expression, which leaves us with a duration of:

$$D_{\text{insurance}} = \frac{\overline{a}_x}{\overline{A}_x}(1 - rD_{\text{annuity}}) \tag{19}$$

Another way to look at this is by explicitly recognizing that:

$$\frac{\partial}{\partial r}\overline{A}_x = \int_0^\infty \frac{\partial}{\partial r}e^{-rt}f_x(t)dt = -\int_0^\infty te^{-rt}f_x(t)dt \tag{20}$$

since we are allowed to interchange the integral and derivatives signs. We are left with a similar "mess" as we had in the previous chapter when we attempted to compute duration for the annuity.

The Duration Value $D = -\frac{\partial}{\partial r}\overline{A}_x / \overline{A}_x$			
Net Single Premium for Life Insurance			
Starting at:	$r = 4\%$	$r = 6\%$	$r = 8\%$
Age 55	22.825 yrs	20.512 yrs	18.209 yrs
Age 65	15.753 yrs	14.304 yrs	12.948 yrs
Age 75	9.912 yrs	9.159 yrs	8.446 yrs
Age 85	5.534 yrs	5.22 yrs	4.927 yrs
Table #6: Source IFID Centre Calculations			
GoMa Mortality $m = 86.34$, $b = 9.5$			

12 Lapsing a Policy

Imagine that for some reason a fraction of this group "do not qualify" to receive the death benefit of \$1. I will model the rate at which individuals leave the insured group using a hazard rate denoted by $\eta(t)$, with the usual proviso that $H_x(t)$ denotes the CDF and $h_x(t)$ denotes the PDF of the remaining "un-lapsed time" random variable, so that $H_x(t) := \Pr[\mathbf{L} \leq t]$. In this case, the lapse-adjusted net single premium would be:

$$\begin{aligned} \left({}_u\overline{A}^{\eta}_{x:\tau}\right) &:= \int_u^{\tau} e^{-rt} f_x(t)\left(1 - H_x(t)\right) dt \qquad (21) \\ &= \left({}_u\overline{A}_{x:\tau}\right) - \int_0^{\infty} e^{-rt} f_x(t) H_x(t) dt \end{aligned}$$

where – with my sincere apologies – the new superscript η on the A_x denotes the fact that we are working with a lapse curve $\eta(t)$. The probability of being un-lapsed is $1 - H_x(t)$, which is akin to the probability of being un-dead. Therefore, the only difference between the integrand in equation (21) and the conventional and expected $e^{-rt} f_x(t)$, is the additional term $1 - H_x(t)$. It should come as no surprise to the reader that $\left({}_u\overline{A}^{\eta}_{x:\tau}\right) \leq \left({}_u\overline{A}_{x:\tau}\right)$, since a fraction of the people paying the (adjusted) net single premium will not be collecting their death benefit.

At this point we must "hand" the mathematics over to the rules of calculus and integration by parts. The final expression for $({}_u\overline{A}^{\eta}_{x:\tau})$ will depend on the precise structure of the $H_x(t)$ function. The easiest possible case is when the un-lapsed time random variable has a constant instantaneous hazard rate η, which leads to the CDF of $H_x(t) = 1 - e^{\eta t}$ and a modified net single premium of:

$$\left({}_u\overline{A}^{\eta}_{x:\tau}\right) := \int_u^{\tau} e^{-(r+\eta)t} f_x(t) dt. \tag{22}$$

In this case the lapse rate η can be absorbed or added into the valuation rate r and the valuation formula for GoMa mortality, for example, can be used with $r + \eta$ instead of just r. And, the process of converting the net single premium into a periodic annual premium would proceed along the same lines. I define the modified pension annuity factor by:

$$\begin{aligned} \left({}_u\overline{a}^{\eta}_{x:\tau}\right) \ : &= \int_u^{\tau} e^{-rt}(1 - F_x(t))(1 - H_x(t))dt \\ &= \int_u^{\tau} e^{-rt} e^{-\int_0^t (\lambda(x+s)+\eta(x+s))ds} dt, \end{aligned} \tag{23}$$

where I have written both $H_x(t)$ and $F_x(t)$ in terms of their primitive definition based on instantaneous hazard rates and mortality forces.

When $\eta(x + t) = \eta$, the instantaneous hazard rate can also be absorbed into the valuation rate r and the valuation equations proceed as before. Of course, when $\eta(x + s)$ is a more complicated function of time, there is no choice but to "roll up your sleeves" and compute the integral in equation (22) and equation (23) by brute force. For example, when $m = 86.34$, $b = 9.5$ and $\lambda = 0$ under a $r = 6\%$ valuation rate, the NSP for an $x = 50$ year-old is $\overline{A}_{45:20}/\overline{a}_{45:20} = 0.06257$ per year for a \$1 death benefit. This translates into $\$625.7$ per \$100,000 death benefit, or $625.7/12 = 52.\,14$ per month, which is consistent with the numbers in Table #6. Now, if I assume that in each instant $0.05dt$ of the surviving group "lapse" and stop paying their insurance premiums, then I can replace $r = 6\%$ by $r + \eta = 11\%$ in the valuation equation for GoMa mortality. This leads to $\$530.38$ per year which is $530.38/12 = 44.\,20$ per month, which is reduction of approximately 20% in the required insurance premium.

To recap, the lapse-adjusted annual premium for a τ-year term insurance policy is:

$$\frac{e^{-(r+\eta)\tau}F_x(\tau) - r\overline{a}_{x:\tau} - e^{-(r+\eta)\tau} + 1}{\overline{a}_{x:\tau}}, \tag{24}$$

where the valuation rate for all pension annuity calculations must be replaced by $r + \eta$ and the deferred pension annuity factor $\overline{a}_{x:\tau}$ can be computed via $\overline{a}_x - ({}_\tau\overline{a}_x)$, both of which are easily available in analytic format.

Monthly Premiums for $100,000 of Life Insurance			
50 year-old under varying "lapse rate" assumptions			
Term	$\eta = 3\%$	$\eta = 5\%$	$\eta = 10\%$
5 yrs	24.68	24.57	24.30
10 yrs	31.24	30.71	29.45
20 yrs	47.15	44.20	38.13
Table #7: Source: The IFID Centre			
Note: $m = 86.34$, $b = 9.5$ and $r = 6\%$			

13 Following a book of Insurance Policies

See spreadsheet....