

Lecture #8:
**A Closer Look at Annuities, Longevity
Insurance and Sustainable Spending**

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1 Longevity Insurance Tontine

- The one-year survival probability for a 95 year-old female is approximately 80%.
- Imagine five females place \$100 each in a "tontine pool".
- They invest the \$500 in a one-year bank deposit paying 5% interest.
- At the end of the year the (expected) surviving four split the \$525 investment.
- Each (expected) survivor gets \$131.25, which is 31.25% interest.
- The extra 26.25% above the risk-free 5% is called mortality credits.
- This is at the heart of all pensions and life annuities.

2 Algebra of Fixed and Variable Tontines

The Algebra of a Fixed Tontine		
Investment	End-of-Year Payoff:	
Now	Alive	Dead
\$100 (Non Tontine)	$100(1 + R)$	$100(1 + R)$
\$100 (Tontine)	$\frac{100}{{}_1p_x}(1 + R)$	0

Table 11.1

Investment Returns from Fixed Tontines			
	Survival	Payoff:	Mortality Credits (b.p.)
Age	$({}_1p_x)\%$	$\frac{\$100}{({}_1p_x)}(1 + R)$	$10000\left(\frac{1}{({}_1p_x)} - 1\right)(1 + R)$
30	99.97%	\$105.03	3.1
50	99.76%	\$105.25	25.4
60	99.31%	\$105.73	73.1
65	98.83%	\$106.24	124.0
70	98.03%	\$107.11	210.8
75	96.69%	\$108.59	359.3
80	94.46%	\$111.15	615.3
85	90.81%	\$115.63	1,062.6
90	84.94%	\$123.61	1,861.0
Table 11.2 GoMa mortality $m = 86.34$, $b = 9.5$, $R = 5\%$			

The Algebra of a Variable Tontine		
Investment	End-of-Year Payoff:	
Now	Alive	Dead
\$100 (Non Tontine)	$100(1 + \mathbf{X})$	$100(1 + \mathbf{X})$
\$100 (Tontine)	$\frac{100}{{}_1p_x}(1 + \mathbf{X})$	0

Table 11.3

3 Utility and Annuities

Let us think of (pension) annuities as longevity insurance against living, "too long". What is this insurance worth?

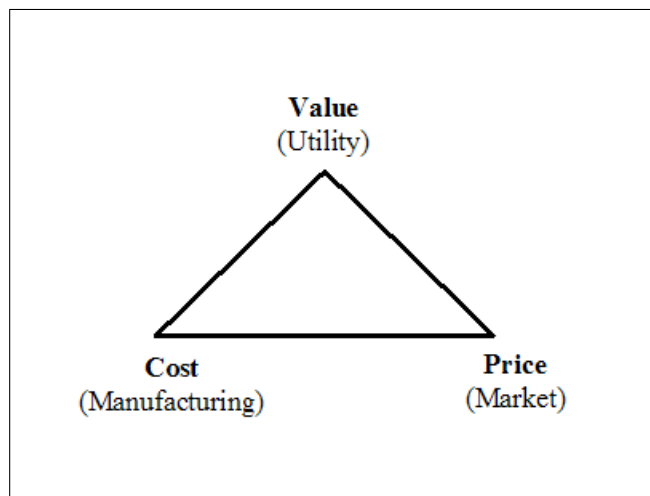


Figure 9.1

In general, there are (at least) three numbers one can use to describe the "worth" of a life-contingent cash-flow.

4 The Utility Function and Insurance

A very common utility function is:

$$U(w) = \frac{1}{1-\gamma} w^{(1-\gamma)}, \quad (1)$$

where U is the utility of wealth, w is the initial net worth and the γ coefficient reflects the individual's personal economic risk aversion level. Note that when $\gamma \rightarrow 1$, equation 1 collapses to a logarithmic utility function $\ln(w)$. Further, when $\gamma = 0$, this individual is risk neutral. When $\gamma > 0$, he or she is increasingly risk averse and in the case that $\gamma < 0$, we have risk loving behavior. Take the following derivatives:

$$U'(w) = w^{-\gamma}, U''(w) = -\gamma w^{-(\gamma+1)},$$

which are the first and second derivatives of the utility function, respectively. I define the Arrow-Pratt measure of relative risk aversion (RRA) as:

$$-w \frac{U''(w)}{U'(w)} = -w \frac{-\gamma w^{-(\gamma+1)}}{w^{-\gamma}} = \gamma$$

The higher the RRA, the more one is willing to pay for insurance. Plugging in various values of $\gamma = 3, 4, 5$ into the utility function, we arrive at the following values:

$$U(w) = \frac{1}{-2w^2}, U(w) = \frac{1}{-3w^3}, U(w) = \frac{1}{-4w^4}$$

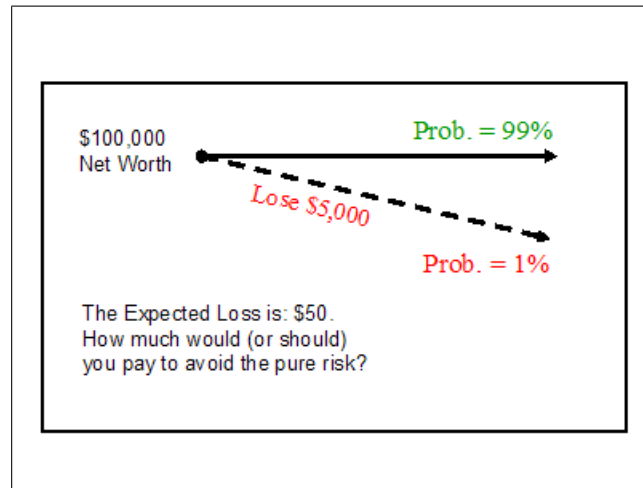


Figure 9.2

Think of insurance against a potential \$5,000 loss with a 1% probability. The pure premium you would have to pay, in theory, to insure against this risk is \$50. But, how much of your wealth are you *willing* to give up, to avoid the $L = 5,000$ magnitude loss with $p = 0.01$ probability, assuming that your current wealth is \$100,000? Namely, we need to solve for the the subjective insurance premium I_γ under a coefficient of relative risk aversion (CRRA) γ , that satisfies:

$$U(w - I_\gamma) = E[U(\tilde{w})] = pU(w - L) + (1 - p)U(w),$$

which, in our example, using equation 1, expands to:

$$\begin{aligned} & \frac{1}{1-\gamma}(100000 - I_\gamma)^{1-\gamma} \\ = & (0.01)\frac{(100000 - 5000)^{1-\gamma}}{1-\gamma} + (0.99)\frac{(100000)^{1-\gamma}}{1-\gamma}. \end{aligned}$$

The first term on the right-hand-side (RHS) is the utility of wealth upon loss and the second term on the RHS is the utility of wealth from no loss. Thus, the utility of your after-loss wealth is a probability weighted and risk aversion adjusted average of the two conditions. Substituting in values of $\gamma = 1, 2, 3$ leads to the solutions: $I_3 = \$53.97$ for $\gamma = 3$, $I_2 = \$52.60$ for $\gamma = 2$ and $I_1 = \$51.28$ for $\gamma = 1$ (using the $\ln(w)$ function). Notice the mark-up above the fair actuarial premium, as the risk aversion increases.

Thus, to recap, in Table 9.1, which displays a spectrum of λ values and the corresponding subjective premiums, the positive relationship between the two variables, is clear.

The relationship between γ and I	
γ	I
11.0	66.772
5.0	56.853
4.5	56.114
4	55.388
3.5	54.674
3.0	53.972
2.5	53.282
2.0	52.603
1 (log utility)	51.28
0.50	50.634
0.00	50.00
-1.00	48.761
Table 9.1	

5 Utility and Lifetime Uncertainty

I now provide a simple two-period example that illustrates the gains in utility from having access to a life annuity market. Assume we have \$1, which must be consumed during the next two periods. The consumption, denoted by C_1 and C_2 , takes place at the end of the period. There is a p_1 probability that the individual will survive to and consume at the end of the first period, and a p_2 probability of surviving to (consuming at) the end of the second period. The periodic interest rate is denoted by R . The objective is to maximize the discounted utility of consumption.

$$\max_{\{C_1, C_2\}} E[U] = \frac{p_1}{1 + \rho} \ln[C_1] + \frac{p_2}{(1 + \rho)^2} \ln[C_2], \quad (2)$$

$$\text{st} \quad 1 = \frac{C_1}{1 + R} + \frac{C_2}{(1 + R)^2}, \quad (3)$$

where ρ is the subjective discount rate. The solution to this consumption-investment problem is obtained by creating the Lagrangian

$$\max_{\{C_1, C_2, \lambda\}} L = \frac{p_1}{1 + \rho} \ln[C_1] + \frac{p_2}{(1 + \rho)^2} \ln[C_2] \quad (4)$$

$$+ \lambda \left(1 - \frac{C_1}{1 + R} - \frac{C_2}{(1 + R)^2} \right) \quad (5)$$

The first order condition is:

$$\begin{aligned} \frac{\partial L}{\partial C_1} &= \frac{p_1}{(1+\rho)C_1} & -\frac{\lambda}{(1+R)} &= 0 \\ \frac{\partial L}{\partial C_2} &= \frac{p_2}{C_2(1+\rho)^2} & -\frac{\lambda}{(1+R)^2} &= 0 \\ \frac{\partial L}{\partial \lambda} &= -\frac{C_1}{(1+R)} - \frac{C_2}{(1+R)^2} & +1 &= 0 \end{aligned} \quad (6)$$

Solving the system of three equations and three unknowns, I obtain the optimal values for the choice variables:

$$C_1^* = \frac{p_1(\rho R + R + \rho + 1)}{p_2 + p_1\rho + p_1}, \quad C_2^* = \frac{p_2(1 + 2R + R^2)}{p_2 + p_1\rho + p_1} \quad (7)$$

The optimal consumption, in the absence of annuities, is given by equation (7). The ratio of consumption between period one and period two, is: $C_1^*/C_2^* = p_1(1 + \rho)/p_2(1 + R)$. When the subjective discount rate is equal to the interest rate ($\rho = R$), then $C_1^*/C_2^* = p_1/p_2$, which is the ratio of the survival probabilities, and is strictly less than one. Stated differently, the individual consumes less at higher ages. In fact, this result can be generalized to a multiperiod setting.

However, in the presence of an actuarially fair life annuity market, the budget constraint in equation (3) must change to reflect the probability adjusted discount factor. This greatly expands the opportunity set for the consumer, and, will increase the utility.

The optimization problem is now:

$$\max_{\{C_1, C_2\}} E[U] = \frac{p_1}{1 + \rho} \ln[C_1] + \frac{p_2}{(1 + \rho)^2} \ln[C_2], \quad (8)$$

$$\text{st} \quad 1 = \frac{p_1 C_1}{1 + R} + \frac{p_2 C_2}{(1 + R)^2}, \quad (9)$$

The Lagrangian becomes:

$$\max_{\{C_1, C_2, \lambda\}} L = \frac{p_1}{1 + \rho} \ln[C_1] + \frac{p_2}{(1 + \rho)^2} \ln[C_2] \quad (10)$$

$$+ \lambda \left(1 - \frac{p_1 C_1}{(1 + R)} - \frac{p_2 C_2}{(1 + R)^2} \right) \quad (11)$$

The first order condition is:

$$\begin{aligned} \frac{\partial L}{\partial C_1} &= \frac{p_1}{C_1(1+\rho)} & -\frac{\lambda p_1}{(1+R)} &= 0 \\ \frac{\partial L}{\partial C_2} &= \frac{p_2}{C_2(1+\rho)^2} & -\frac{\lambda p_2}{(1+R)^2} &= 0 \\ \frac{\partial L}{\partial \lambda} &= -\frac{p_1 C_1}{(1+R)} - \frac{p_2 C_2}{(1+R)^2} + 1 & &= 0 \end{aligned} \quad (12)$$

The optimal consumption is denoted by C_1^{**} , C_2^{**} , and is equal to:

$$C_1^{**} = \frac{\rho R + R + \rho + 1}{p_2 + p_1 \rho + p_1}, \quad C_2^{**} = \frac{1 + 2R + R^2}{p_2 + p_1 \rho + p_1} \quad (13)$$

The important point to notice is that $C_1^{**} = C_1^*/p_1$ and $C_2^{**} = C_2^*/p_2$, which implies that the optimal consumption is greater in both periods, in the presence of life annuities. Specifically, at time zero, the individual would purchase a life annuity that pays C_1^{**} at time 1 and C_2^{**} at time 2. The present value of the two life

annuities – as per the budget constraint – is one dollar. In this case, the ratio of consumption between period one and period two, is: $C_1^*/C_2^* = (1 + \rho)/(1 + R)$. When the subjective discount rate is equal to the interest rate ($\rho = R$), then $C_1^*/C_2^* = 1$, which is the ‘smoothing’ effect of annuities, discussed above.

Here is an numerical example which should help illustrate the model. Let $R = \rho = 10\%$, and let $p_1 = 0.75$ and $p_2 = 0.40$. The individual has a 75% chance of surviving to the end of the first period, and a 40% chance of surviving to the end of the second period. Hence, according to equation (7), the optimal consumption is: $C_1^* = 0.741$ and $C_2^* = 0.395$ in the absence of annuities. The maximum utility is $EU^* = -0.5115$. However, in the presence of life annuities, the optimal consumption becomes $C_1^{**} = 0.987$ and $C_2^* = 0.987$ with a maximal utility of $EU^* = -0.01247$, which is clearly greater than the no annuity case. To get a sense of the benefit from annuitizing, if one solves equation (10), with a budget constraint equal to 0.61, instead of 1, the optimal annuitized consumption would be $C_1^{**} = 0.603$ and $C_2^{**} = 0.603$. In this case, the maximal utility would be the same as with the no annuity case. Stated differently, if one was to take away 0.39 from the individual, but give them access to a fairly priced life annuity, the utility would be the same.

6 Asset Allocation with Tontines

Imagine a situation in which you have $W_0 = \$100$ that you would like to "allocate" between a "safe bonds" fund yielding an interest rate of R during the next year and a "risky stocks" fund which yields or returns a random \mathbf{X} during the next year. Assume that your allocation proportion is denoted by the symbol θ , which can range from $\theta = 0\%$ to $\theta = 100\%$ allocated to the risky investment fund. In general, if W_0 denotes your initial investment or wealth, at the end of the year you will have a total of:

$$\mathbf{W}_1 = (W_0 (\theta(1 + \mathbf{X}) + (1 - \theta)(1 + R))). \quad (14)$$

If the expected investment return from the risky stock is denoted by $E[\mathbf{X}] = \nu$, the volatility or standard deviation of this return is $SD[\mathbf{X}] = \sigma$ and the investment return \mathbf{X} itself is normally distributed, then the end-of-year portfolio value will also be normally distributed with a mean value of:

$$E[\mathbf{W}_1] = W_0 (\theta(1 + \nu) + (1 - \theta)(1 + R)) \quad (15)$$

$$SD[\mathbf{W}_1] = W_0 \theta \sigma \quad (16)$$

For example, in the above mentioned case, where $R = 5\%$ and $\theta = 60\%$, if the risky stock satisfies $\nu = 11\%$ and $\sigma = 20\%$, then $E[\mathbf{W}_1]/W_0 - 1 = 8.6\%$ and the standard deviation is $SD[\mathbf{W}_1]/W_0 = (0.6)(0.2) = 12\%$. Note that I subtracted one from the ratio $E[\mathbf{W}_1]/W_0$ to convert the total return into a rate of return.

The objective function. Assume you want to maximize expected returns:

$$\max_{\theta} E[\mathbf{W}_1] \quad (17)$$

subject to the constraint that:

$$\Pr[\mathbf{W}_1 \leq W_0] \leq \varepsilon. \quad (18)$$

Recall that the probability a standard normal random variable "takes on" a value less than or equal to c , can be denoted and is equal to:

$$\Phi(c) = \int_{-\infty}^c \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz. \quad (19)$$

And, the probability that a non-standard normal random variable will take on a value less than or equal to c is equal to $\Phi((c-\nu)/\sigma)$, where ν is the mean and σ is the standard deviation.

For any given value of θ , the probability the portfolio \mathbf{W}_1 is worth less than its initial value W_0 is:

$$\Phi\left(\frac{W_0 - W_0(\theta(1 + \nu) + (1 - \theta)(1 + R))}{W_0\theta\sigma}\right), \quad (20)$$

We are looking for the largest value of θ – which will become our optimum θ^* – such that the probability of shortfall (loss) is exactly equal to ε . After some basic cancellations and simple algebra, we can "invert" the function $\Phi(\cdot)$ and search for the largest value of θ such that:

$$\frac{1}{\theta\sigma} - \left(\frac{1 + \nu}{\sigma} + \frac{1 + R}{\theta\sigma} - \frac{1 + R}{\sigma}\right) = \Phi^{-1}(\varepsilon), \quad (21)$$

where $\Phi^{-1}(\varepsilon)$ denotes the inverse of the normal CDF evaluated at ε . For example, $\varepsilon = 0.01$ leads to $\Phi^{-1}(0.01) = -2.326$, while $\Phi^{-1}(0.10) = -1.281$ and obviously $\Phi^{-1}(0.5) = 0$. This leads to:

$$-\left(\frac{\nu - R}{\sigma}\right) - \frac{R}{\theta\sigma} = \Phi^{-1}(\varepsilon), \quad (22)$$

which – by isolating the choice variable θ – finally leads us to:

$$\theta^* = \frac{R}{-\sigma\Phi^{-1}(\varepsilon) - (\nu - R)}. \quad (23)$$

There are a number of technical conditions for this to work: first and foremost: $-\sigma\Phi^{-1}(\varepsilon) - (\nu - R) > 0$, which means that $(\nu - R)/\sigma < -\Phi^{-1}(\varepsilon)$.

Now let me investigate the same problem, when the asset allocation decision takes place within the above-mentioned tontines. In this case the R variable is replaced with $(1 + R)/({}_1p_x) - 1$, the ν variable is replaced with $(1 + \nu)/({}_1p_x) - 1$ and the standard deviation σ of the "risky asset" is replaced by $\sigma/({}_1p_x)$.

Another way to think about this is by examining the tontine-adjusted portfolio mean and standard deviation via:

$$E[\mathbf{W}_1^{\text{Tontine}}] = W_0 (\theta(1 + \nu) + (1 - \theta)(1 + R)) / ({}_1p_x) \quad (24)$$

$$SD[\mathbf{W}_1^{\text{Tontine}}] = W_0 \theta \sigma / ({}_1p_x). \quad (25)$$

The optimization problem remains the same, except that the probability constraint must now be written as:

$$\Phi\left(\frac{W_0 - W_0 (\theta(1 + \nu) + (1 - \theta)(1 + R)) / ({}_1p_x)}{W_0 \theta \sigma / ({}_1p_x)}\right) \leq \varepsilon. \quad (26)$$

Going through similar algebra as before – and canceling the $({}_1p_x)$ wherever possible, I am left with the problem of locating the largest value of θ such that:

$$\frac{({}_1p_x)}{\theta \sigma} - \left(\frac{1 + \nu}{\sigma} + \frac{1 + R}{\theta \sigma} - \frac{1 + R}{\sigma} \right) = \Phi^{-1}(\varepsilon), \quad (27)$$

which can be simplified to:

$$\frac{({}_1p_x) - (1 + R)}{\theta} - (\nu - R) = \sigma \Phi^{-1}(\varepsilon) \quad (28)$$

which then leads to:

$$\theta^{**} = \frac{R + (1 - ({}_1p_x))}{-\sigma\Phi^{-1}(\varepsilon) - (\nu - R)} \quad (29)$$

Table 11.4 provides numerical estimates under a variety of loss tolerance levels ε , both with and without tontines.

I Want to Avoid Losing Any Money			
The Portfolio Mix of Stocks and Bonds			
Loss	Allocation to Stocks		
Tolerance	θ^*	θ^{**} (Age 75 Tontine)	θ^{**} (Age 60 Tontine)
$\varepsilon = 1\%$	12.34%	20.51%	14.04%
$\varepsilon = 5\%$	18.59%	30.90%	21.15%
$\varepsilon = 10\%$	25.47%	42.33%	28.98%
$\varepsilon = 20\%$	46.16%	76.71%	52.53%
$\varepsilon = 25\%$	66.76%	110.95%	75.97%
Table 11.4 $E[X] = 11\%$, $SD[X] = 20\%$, $R = 5\%$			
$({}_1p_{60}) = 99.31\%$, $({}_1p_{75}) = 96.69\%$, $m = 86.34$, $b = 9.5$			

7 A First Look at Self-Annuitization

The basic market pricing definition of a one-dollar per year pension annuity in discrete time, is:

$$a_x = \sum_{i=1}^{\infty} \frac{{}_i p_x}{(1 + R)^i}, \quad (30)$$

where R denotes the effective annual valuation rate used by the insurance company to discount cash flows, ${}_i p_x$ denotes the conditional probability that an individual aged (x) will attain age $(x + i)$.

Now, imagine that instead of purchasing a pension annuity the retiree decides to delay purchasing the life annuity for one year, until age $(x + 1)$. Now, in order to afford the exact same life annuity stream in one year, the annual investment return G earned by the retiree, must satisfy the following inequality:

$$a_x(1 + G) - 1 \geq a_{x+1}. \quad (31)$$

The condition for beating the rate of return from the annuity, over one year, is:

$$G \geq \frac{a_{x+1}}{a_x} + \frac{1}{a_x} - 1. \quad (32)$$

The right-hand side of equation (32) is the threshold annual investment return necessary for what I like to call a "successful" deferral decision. In general, using the actuarial identity:

$$({}_t p_{x+n}) = \frac{({}_{n+t} p_x)}{({}_n p_x)}, \quad (33)$$

I can re-write a_{x+1} in terms of a_x and then re-write the condition for beating the rate of return on the annuity, using equation (32), as:

$$G \geq \frac{1 + R}{({}_1 p_x)} - 1. \quad (34)$$

Equation (34) should be recognized as the investment return as well as the "mortality credit" from the tontine and is crucial to my main thesis.

8 Implied Longevity Yield (ILY)

What is the required return to beat the pension (life) annuity over a multiperiod framework?

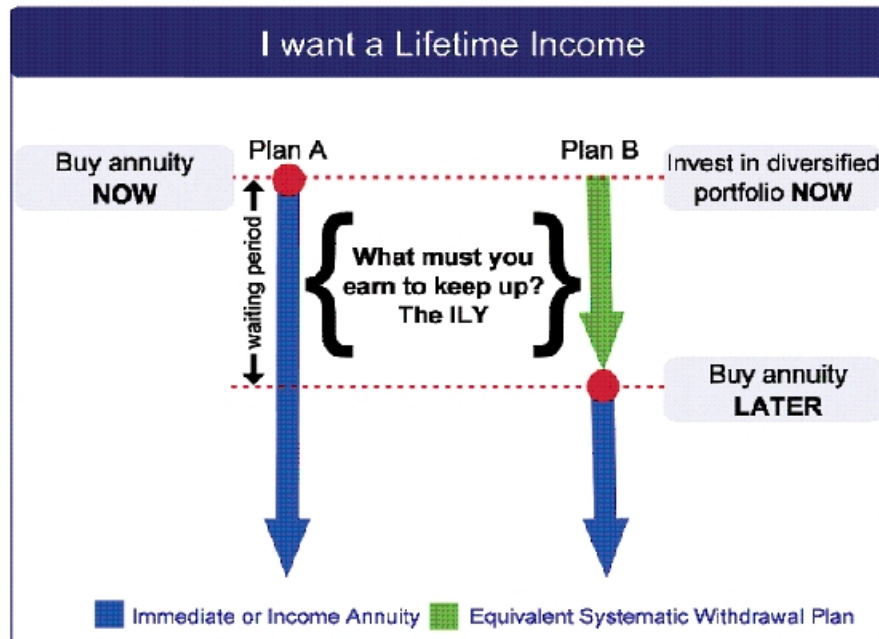


Figure 11.1

9 Analytics of ILY

Assume a hypothetical retiree who has W_0 dollars in marketable wealth. If this individual were to annuitize – i.e. to convert a stock of wealth W into a lifetime flow – he or she would be entitled to W/a_1 per annum for life, where a_1 is short-hand notation for the relevant pension annuity factor at the relevant age. If, in contrast, the retiree decided to forgo the purchase of the life annuity and instead self-annuitized – by investing the funds at a "fixed" force of interest denoted by g and consuming in continuous time at the annuity rate W/a_1 – the wealth dynamics would satisfy the Ordinary Differential Equation (ODE):

$$dW_t = \left(gW_t - \frac{W_0}{a_1} \right) dt, \quad W_t \geq 0 \quad (35)$$

The solution to the ODE in equation (35) is:

$$W_t = \left(W_0 - \frac{W_0}{ga_1} \right) e^{gt} + \frac{W_0}{ga_1}, \quad W_t \geq 0 \quad (36)$$

where g can always be selected so that $W_t > 0$ for all values of t . But, if this investment portfolio is to contain enough funds to purchase the *same exact annuity flow* at age $x + u$, the following relationship must hold:

$$\frac{W_0}{a_1} a_2 = \left(W_0 - \frac{W_0}{ga_1} \right) e^{gu} + \frac{W_0}{ga_1}, \quad (37)$$

where a_2 is short-hand notation for the relevant pension annuity

factor at age $(x + u)$.

We are then searching for a value of g that equates both sides. Finally, dividing by W_0 and multiplying by a_1 , we are left with:

$$a_2 - \left(a_1 - \frac{1}{g} \right) e^{gu} - \frac{1}{g} = 0. \quad (38)$$

The value of g^* that solves the above equation is defined as the implied longevity yield.

Example: A 65-year-old male is quoted an average monthly payout of \$678.22 per initial premium of \$100,000 with a 10-year payment certain period. The continuous-time annuity factor is approximated as $100000/(12 \times 678.216) = 12.2871$ which is $a_1 = 12.2871$ per \$1-for-life using our notation. On the same exact date, a 75-year-old is quoted an average monthly payout of \$977 per premium of \$100,000 with a zero-year payment certain period. The annuity factor is $100000/(12 \times 975.904) = 8.5391$ which is $a_2 = 8.5391$ per \$1-for-life.

What Does \$100,000 in Premium Buy?			
Monthly Income from Immediate Annuity			
(Average of Best Companies in the U.S.)			
	0 yr p.c.	10 yr p.c	20 yr p.c.
Age 60m	\$582	\$569	\$536
Age 60f	\$546	\$539	\$519
Age 70m	\$741	\$689	
Age 70f	\$674	\$644	
Age 80m	\$1075		
Age 80f	\$967		
Table 11.5 Source: CANNEX Financial Exchanges			
March, 2005			

I am searching for the g that the 65-year-old would have to earn on their discretionary investment portfolio to beat the return from the annuity, but still consume the exact same income on an ongoing basis. The situation we are faced with is equation (38) with $u = 10$ years, $x = 65$ and g being the unknown return variable.

$$8.5391 - \left(12.2871 - \frac{1}{g}\right) e^{10g} - \frac{1}{g} = 0 \quad (39)$$

The solution must be computed numerically due to the non-linearity of the equation, and is $g^* = 0.0590$ which is an ILY value of 5.90%. As stated earlier, the 65-year-old male would have to earn 5.90% per annum each year for the next 10 years to beat the return from the annuity. Ergo, the value of the ILY on the date in question is 5.90% for males.

What Does \$569 Per Month Cost a Male?			
Monthly Income from Immediate Annuity			
(Average of Best Companies in the U.S.)			
	0 yr p.c.	10 yr p.c	20 yr p.c.
Age 60m	\$97,816	\$100,000	\$106,204
Age 70m	\$76,892	\$82,675	
Age 80m	\$52,984		
Implied Longevity Yield (Age x (p.c.) to Age y (p.c.))			
Age 60 (10) to Age 70 (0)			5.06%
Age 70 (10) - Age 80 (0)			5.58%
Age 60 (20) - 80 (0)			4.97%
Table 11.6 Source: CANNEX Financial Exchanges, 03/2005			
Treasury Yield Curve Rates: 10yr - 4.38%, 20yr - 4.80%			

What Does \$539 Per Month Cost a Female?			
Monthly Income from Immediate Annuity			
(Average of Best Companies in the U.S.)			
	0 yr p.c.	10 yr p.c.	20 yr p.c.
Age 60m	\$98,712	\$100,000	\$103,874
Age 70m	\$80,054	\$83,756	
Age 80m	\$55,779		
Implied Longevity Yield (Age x (p.c.) to Age y (p.c.))			
Age 60 (10) to Age 70 (0)			4.93%
Age 70 (10) - Age 80 (0)			5.18%
Age 60 (20) - 80 (0)			4.86%
Table 11.7 Source: CANNEX Financial Exchanges, 03/05			
Treasury Yield Curve Rates: 10yr - 4.38%, 20yr - 4.80%			

Should an 80 Year-Old Annuitize?				
Age, Gender, Period Certain	80,M,5	85,M,0	80,F,5	85,F,0
Monthly Income	\$995	\$1,352	\$917	\$1,231
ILY	7.58%		6.71%	
Table 11.8 Source: CANNEX Financial Exchanges, 03/05.				
Treasury Yield Curve Rates: 5 yr - 4.02%				

10 Generalized Mortality Credits

There is a close relationship between our ILY values and *actuarial mortality credits*. To see this connection explicitly, I analyze the simplest possible case of annuity pricing, namely when the valuation rate is constant at r and the force of mortality is constant at $\lambda(x) = \lambda$ at all ages, and all annuities are life-only with no guarantee period. In this case, annuity pricing equation collapses to:

$$\bar{a}_x = \int_0^{\infty} e^{-(r+\lambda)s} ds = \frac{1}{\lambda + r}, \quad (40)$$

regardless of the age x . Using our short-hand notation both a_1 and a_2 are therefore equal to $(r + \lambda)^{-1}$ since exponential mortality (and a constant hazard rate) is synonymous with no aging.

The fundamental equation for the ILY is then:

$$\frac{1}{\lambda + r} - \left(\frac{1}{\lambda + r} - \frac{1}{g} \right) e^{gu} - \frac{1}{g} = 0, \quad (41)$$

whose solution is precisely $g = r + \lambda$ regardless of the value of u . In other words, the self-annuitization strategy must earn (and the ILY value must be) at least λ above the pricing rate r in order to purchase the same annuity income flow in the future.

11 Numerical Value of ILY

We can approximate the exponential term e^{gu} over small values of g with the quadratic form $(1 + gt + 0.5(gt)^2)$. Using this approximation and then collecting terms, the implied longevity yield is the value of g that solves:

$$-\left(\frac{1}{2}a_1u^2\right)g^2 + \left(\frac{1}{2}u^2 - a_1u\right)g + (a_2 + u - a_1) \approx 0. \quad (42)$$

The solution to this quadratic equation in g is:

$$g^* \approx \frac{(u - 2a_1) + \sqrt{u^2 + 4a_1(u + 2a_2 - a_1)}}{2ua_1}. \quad (43)$$

In our earlier case (Male 65) for which $a_1 = 12.2871$ and $a_2 = 8.5391$ the exact value of the ILY is $g^* = 5.900\%$ using the NeRa method. According to the approximation in equation (43) we obtain:

$$\begin{aligned} g^* &\approx \frac{(10 - 24.5742) + \sqrt{100 + 49.1484(10 + 17.0782 - 12.2871)}}{2(10)(12.2871)} \\ &= 0.05771, \end{aligned}$$

which is an ILY value of 5.771%, a mere 12 basis points lower than the true value. Or, in the joint-life case mentioned above, the precise value of the ILY was $g^* = 5.21\%$ while the approximate value is $g^* = 5.14\%$. Note. The quadratic approximation consistently underestimates the true value of g^* by between 10 and 20 basis points.

12 Self-Annuitization (Delay) for Ever?

What if you start with the sum of a , and self-annuitize for ever? The W_t process will now obey the ordinary differential equation (ODE),

$$dW_t = (\mu W_t - 1) dt, \quad W_0 = a, \quad W_t \geq 0, \quad (44)$$

where μ is the arithmetic (continuously compounded) mean return. Without any loss of generality, we can define this equation up to the point of ruin $W_{t^*} = 0$. The solution to the ODE is:

$$W_t = \begin{cases} \left(a - \frac{1}{\mu}\right) e^{\mu t} + \frac{1}{\mu} & t < t^* \\ 0 & t \geq t^* \end{cases}, \quad (45)$$

where t^* is the time of ruin. This value can be obtained exactly by solving:

$$\left(a - \frac{1}{\mu}\right) e^{\mu t} + \frac{1}{\mu} = 0 \quad \iff \quad t^* = \frac{1}{\mu} \ln \left[(a\mu - 1)^{-1} \right] \quad (46)$$

Finally, when the initial value of the function/process W_0 is arbitrarily set equal to $a = (\lambda + \mu)^{-1}$, the ruin time t^* can be simplified to:

$$t^* = \frac{1}{\mu} \ln \left[1 + \frac{\mu}{\lambda} \right] \quad (47)$$

And, when $\mu = \lambda$, the value of $t^* = \ln[2]/\lambda$, which is exactly the *median* life span. In the limit, as $\mu \rightarrow 0$, the ruin time is precisely

the life expectancy $1/\lambda$, since

$$\lim_{\mu \rightarrow 0} \frac{1}{\mu} \ln\left[1 + \frac{\mu}{\lambda}\right] = \frac{1}{\lambda} \quad (48)$$

Finally, the probability of not surviving to the point at which W_t hits zero, is:

$$1 - e^{-\lambda t^*} = 1 - e^{-\frac{\lambda}{\mu} \ln\left[1 + \frac{\mu}{\lambda}\right]}. \quad (49)$$

13 Probability of Retirement Ruin

If one is willing to assume lognormal returns in a continuous time setting, under μ, σ parameters and λ exponential mortality, the probability of retirement ruin is:

$$\Pr[SV P > w] = \text{GammaDist}\left(\frac{2\mu + 4\lambda}{\sigma^2 + \lambda} - 1, \frac{\sigma^2 + \lambda}{2} \mid \frac{1}{w}\right) \quad (50)$$

where $\text{GammaDist}(a, \beta \mid \cdot)$ denotes the cumulative distribution function (CDF) of the Gamma distribution evaluated at the parameter pair α, β .

For example, start with an investment (endowment, nest egg) fund containing $w = 20$ that is invested in a fund that is expected to earn $\mu = 0.07$ per annum, with a volatility or standard deviation of $\sigma = 0.20$ per annum. This is a $\nu = 0.05$ value. Assume that a (unisex) 50-year-old with a median remaining lifetime of 28.1 years intends on consuming \$1 after-inflation per annum for the rest of his or her life.

Recall that if the median lifespan is 28.1 years, then by definition, the probability of survival for 28.1 years is exactly 50%, which implies that our “mortality rate” parameter is: $\lambda = \ln[2]/28.1 = 0.0247$. According to equation 50, the so-called probability of retirement ruin is approximately 26.8%. Naturally, different values of initial wealth w , or different values of spending rates $1/w$ will result to different ruin probabilities.

What is the probability your spending is not sustainable?						
Random lifespan & random returns under a fixed spending rate						
Spending Rate (Per \$100):			\$4.0	\$5.0	\$6.0	\$10.0
Age x	Med $x + T$	Mortality λ				
N.A	infinity	0.00%	40.7%	66.7%	84.5%	99.8%
55	83.0	2.48%	10.8%	20.1%	31.2%	72.4%
65	83.9	3.67%	7.0%	13.2%	21.0%	56.4%
70	84.6	4.75%	5.0%	9.5%	15.3%	45.0%
75	85.7	6.48%	3.1%	6.0%	9.9%	31.9%
80	87.4	9.37%	1.7%	3.2%	5.4%	19.1%
Table 10.3 $\mu = 5.0\%$, $\sigma = 10.0\%$						

Notice the impact of higher spending and younger retirement ages.