FINE6860: Lecture #1
Human Financial Life-Cycle

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1 The Next 60 Years of Your Life

Question: *What fraction of your salary must you save during your 30 years of work so that when you retire with your accumulated nest egg, you can generate an equivalent income stream that will last for the remaining 30 years of life?*

2 Future Value of Savings:

$$FV(S,R,N) = \sum_{i=1}^{N} S(1+R)^{(N-i)} \tag{1}$$

Note that $S(1+R)^{(N-i)}$ when $i = N$, is exactly S.

Now, using no more than basic algebra, the right-hand side of equation (1) can be expressed in closed-form (provided $R > 0$) and without a summation sign, by:

$$FV(S,R,N) = S\frac{(1+R)^N - 1}{R} \tag{2}$$

For example, if you save \$1 at the end of each year, for 30 years at a 5% interest, the future value of your savings at retirement will be: $FV(1, 0.05, 30) = 66.439$.The relationship is not linear in the interest rate R. The precise value is $FV(1, 0.10, 30) = 164.49$, which is roughly 150% more wealth at retirement, when you earn $R = 10\%$ versus $R = 5\%$.

3 Present Value of Consumption:

$$PV(C, R, D) = \sum_{j=1}^{D} \frac{C}{(1+R)^j}. \tag{3}$$

Using basic algebra to add-up the series on the right-hand side of equation (3), we arrive at:

$$PV(C, R, D) = C\frac{1-(1+R)^{-D}}{R}, \tag{4}$$

with a similar understanding that $PV(C, 0, D) := C \times D$.

Wealth after 1st Year in Retirement:

$$\text{[End of Year \#1]} = \mathbf{X}(1+R) - 1.$$

The Wealth after 2nd Year in Retirement:

$$\text{[End of Year \#2]} = (\mathbf{X}(1+R) - 1)(1+R) - 1.$$

The Wealth after 3rd Year in Retirement:

$$\text{[End of Year \#3]} = ((\mathbf{X}(1+R) - 1)(1+R) - 1)(1+R) - 1$$

Wealth after Dth Year in Retirement:

$$= \mathbf{X}(1+R)^D - \sum_{j=1}^{D}(1+R)^{D-j}. \tag{5}$$

If your wealth – or portfolio – hits zero precisely at the very end of the Dth year of retirement, then X must satisfy the following equation:

$$\mathbf{X} := \sum_{j=1}^{D} \frac{1}{(1+R)^j}. \tag{6}$$

If your initial retirement nest egg is exactly equal to $\mathbf{X}$, is invested at a rate of $R\%$, and if you consume \$1 each and every year, then you will run out of money at time D. This value of $\mathbf{X}$ is precisely the present value of \$1 consumption during retirement that is displayed and presented in equation (3) and (4).

4 Exchange Rate Between Savings and Consumption

Equating, equation (2) to equation (4) leaves us with:

$$\begin{aligned} FV(S,R,N) &= PV(C,R,D) \\ C &= S\frac{FV(1,R,N)}{PV(1,R,D)}. \end{aligned} \tag{7}$$

I will define the *exchange rate* or ratio between the future value and the present value by the Greek symbol $\alpha := FV(1,R,N)/PV(1,R,D)$

5 A Neutral Replacement Rate

Equation (7) provides us with a relationship between saving S and the desired consumption C, which in this case is $C = W - S$. This leads to:

$$(W - S) = S\alpha \iff S = \frac{W}{1 + \alpha}, \tag{8}$$

where α denotes the "savings to consumption" exchange rate $FV(1, R, N)/PV(1, R, D)$. Now, it all comes down to the interest rate R. When $R = 8\%$ and $N = D = 30$, we obtain $\alpha = 10.063$ and therefore $S = \$4,520$ dollars, according to equation (8).

Notice that by savings $S = \$4,520$ each year, you are left with a net wage of \$45,480. At the same time, the future value of $S = \$4,520$ is $FV(4520, 0.08, 30) \approx \$512000$ and the present value of the net-wage is also $PV(45480, 0.08, 30) \approx \$512,000$ in retirement.

6 Discounted Value of a Life Cycle Savings and Consumption Plan

If we put both of the above ingredients – savings and consumption phase – together into one large equation, then the total discounted value of both stages in the human life-cycle can be expressed as the **Discounted Value of Life-cycle Plan**

$$DVLP(R,S,C,N,D) := \sum_{i=1}^{N+D} \frac{S_i + C_i}{(1+R)^i}, \tag{9}$$

This leads to:

$$DVLP(R,S,C,N,D) = \frac{S(1-(1+R)^{-N})}{R} - \frac{C(1-(1+R)^{-D})}{R(1+R)^N} \tag{10}$$

For example, $DVLP(0.05, 1, 10, 30, 30) = -20.195\,93$. However, if we increase the interest rate to $R = 8\%$, the discounted value of the same plan $(S = 1, C = 10)$ is now $DVLP(0.08, 1, 10, 30, 30) \approx 0$.

7 The Invariance of Inflation

If you enter the labour force with a wage of W_0 at the start of period $i = 0$ and this wage increases each year due to a constant inflation rate denoted by π, then your nominal wage at the start of period i will be equal to and denoted by:

$$W_i = W_0(1+\pi)^i.$$

Likewise, if we let S^π denote the constant inflation-adjusted value of your savings, then at the end of N years you will have saved a total of:

$$\begin{aligned}
&S^\pi(1+\pi)(1+R)^{N-1} \\
&+S^\pi(1+\pi)^2(1+R)^{N-2} \\
&+S^\pi(1+\pi)^3(1+R)^{N-3} \\
&+...S^\pi(1+\pi)^N,
\end{aligned}$$

in nominal terms, where R denotes the nominal interest rate. However, we can decompose this number into a "real" component and an "inflation" component so that we can write the interest rate as:

$$R = (1+R^\pi)(1+\pi) - 1,$$

where R is the nominal rate and $R^* \leq R$ is the real inflation-adjusted rate. Then, a bit of algebra allows to to express the future

value of savings as:

$$S^{\pi}(1+R^{\pi})^{N}\sum_{i=1}^{N}\frac{(1+\pi)^{i}}{((1+R^{\pi})(1+\pi))^{i}},$$

which collapses to the familiar:

$$S^{\pi}\frac{(1+R^{\pi})^{N}-1}{R^{\pi}}$$

The same results will follow when the present value of consumption is computed at retirement. The relevant sum is replaced by:

$$\frac{C^{\pi}(1-(1+R^{\pi})^{-D})}{R^{\pi}}=\sum_{j=1}^{D}\frac{C^{\pi}(1+\pi)^{j}}{((1+R^{\pi})(1+\pi))^{j}}$$

Example: You plan to save \$10,000 in after-inflation dollars, each year for the next 30 years until retirement. Thus, at the end of year #1 you will save $\$10,000(1+\pi)$, and at the end of year #2 you will save $\$10,000(1+\pi)$, etc. These savings will be invested at a real after-inflation rate of 8% per annum. The nominal interest rate will be $(1+\pi)(1+0.08)-1$. Question: What is the value of your retirement savings after 30 years? Answer: If you don't know what π is, then you will not be able to obtain a nominal (pre-inflation) value of your nest egg. But, the

real (after-inflation) value can easily be calculated via:

$$\$1,132,832 = \$10000\frac{(1.08)^{30}-1}{0.08}$$

And, the nominal value will be $1,132,832 \times (1+\pi)$.

In sum, you are entitled to use the exact same equation and methodology to compute the future value of saving at retirement and/or the present value of consumption at retirement, provided that you replace both savings (in dollars) and interest rates (in percent) to after-inflation values.

8 Changing Interest Rates Over Time

When the interest rate is NOT constant from one period to the next equation (8) should be expressed as the **Discounted Value of Life-cycle Plan:**

$$\text{DVLP} = \sum_{i=1}^{N+D}(S_i - C_i)\prod_{j=1}^{i}(1+R_j)^{-1}, \tag{11}$$

where $T = N + D$ to save on notation.

The savings while working…

interest rate	7.0%	9.0%	11.0%	13.0%
0.0%	8.4%	10.8%	13.2%	15.6%
1.0%	11.1%	14.2%	17.4%	20.5%
2.0%	14.5%	18.7%	22.9%	27.0%
3.0%	19.1%	24.6%	30.1%	35.5%
4.0%	25.1%	32.3%	39.5%	46.7%
5.0%	33.0%	42.4%	51.9%	61.3%
7.0%	56.7%	73.0%	89.2%	105.4%
9.0%	97.1%	124.9%	152.6%	180.4%

work years	30
retire years	25

The savings while working…

interest rate	7.0%	9.0%	11.0%	13.0%
0.0%	7.0%	9.0%	11.0%	13.0%
1.0%	9.4%	12.1%	14.8%	17.5%
2.0%	12.7%	16.3%	19.9%	23.5%
3.0%	17.0%	21.8%	26.7%	31.6%
4.0%	22.7%	29.2%	35.7%	42.2%
5.0%	30.3%	38.9%	47.5%	56.2%
7.0%	53.3%	68.5%	83.7%	99.0%
9.0%	92.9%	119.4%	145.9%	172.5%

work years	30
retire years	30

The savings while working…

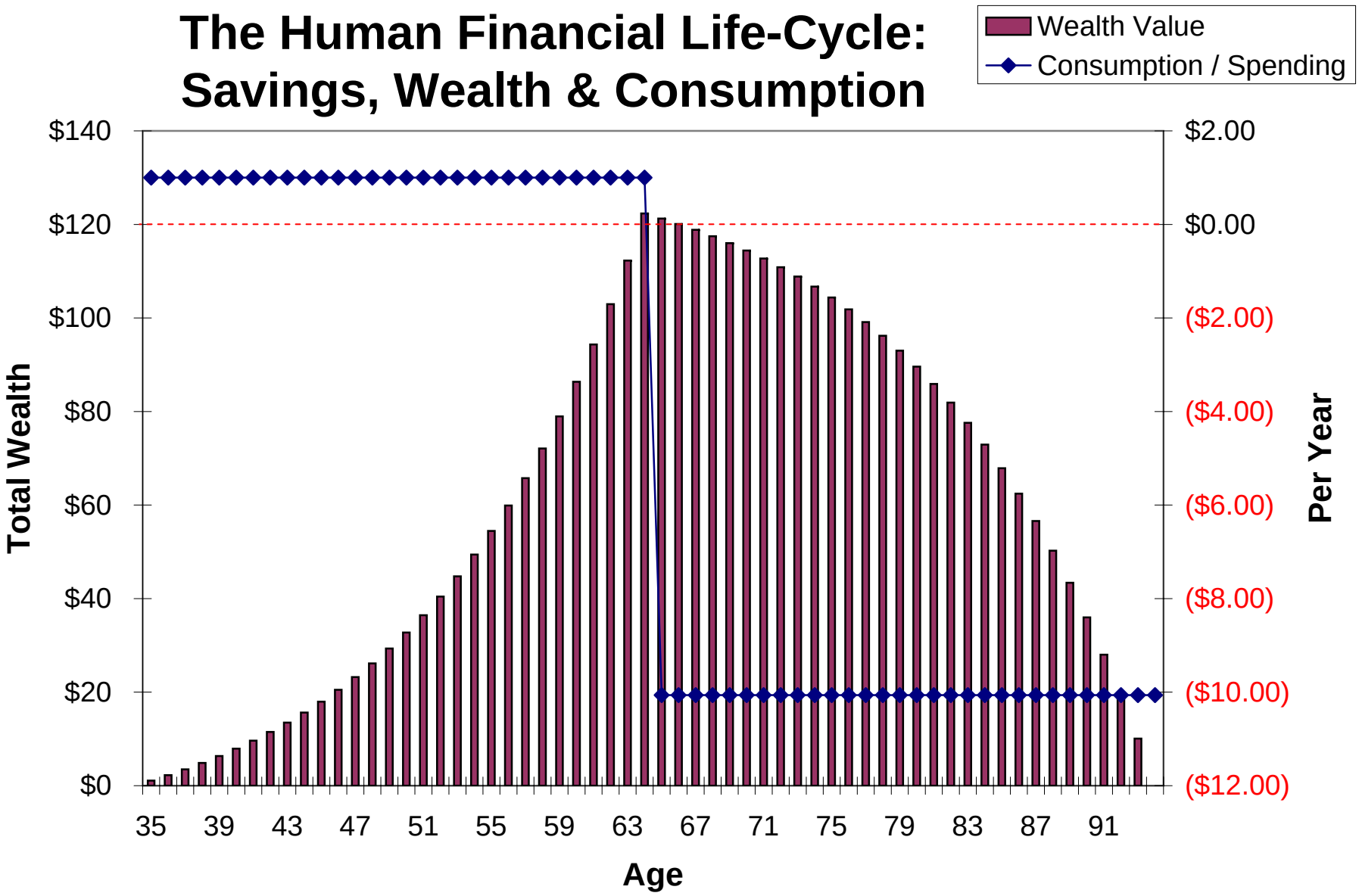
interest rate	7.0%	9.0%	11.0%	13.0%
0.0%	5.3%	6.8%	8.3%	9.8%
1.0%	7.4%	9.5%	11.7%	13.8%
2.0%	10.4%	13.3%	16.3%	19.3%
3.0%	14.4%	18.5%	22.6%	26.8%
4.0%	19.8%	25.5%	31.2%	36.8%
5.0%	27.1%	34.8%	42.6%	50.3%
7.0%	49.6%	63.8%	77.9%	92.1%
9.0%	88.7%	114.0%	139.4%	164.7%

work years	30
retire years	40

$$\sum_{i=1}^{N+D}(C_i + S_i)\prod_{j=1}^{i}(1+R_j)^{-1}$$

Mean	8.00%
Std.Dev	0.00%

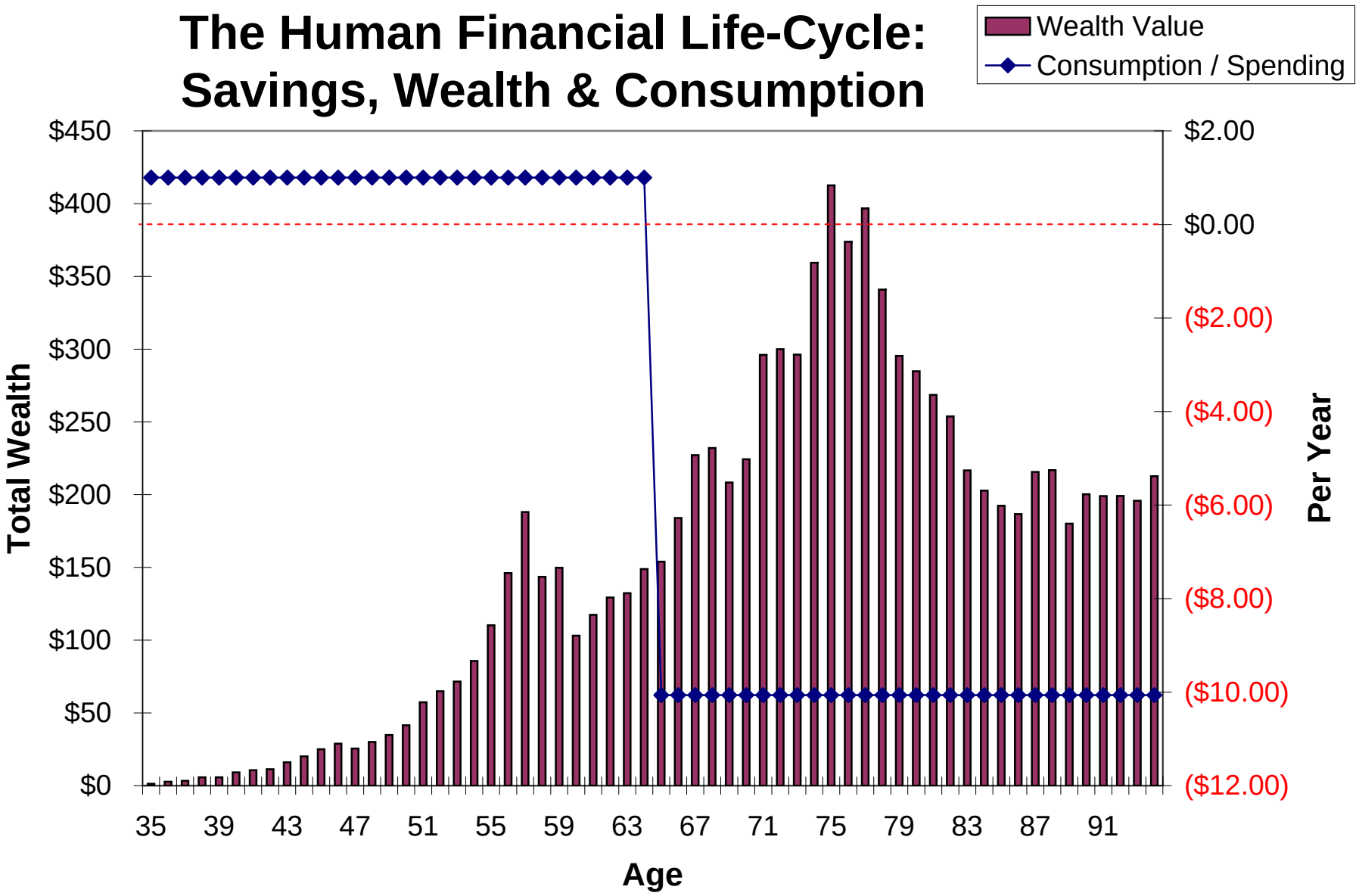
Save / Consume	E.O.Y. Age	Investment Return	DVLP @ 35th Birthday		
			$0.000		
$1.00	35	8.00%	$0.000	$1.08	$1.00
$1.00	36	8.00%	($1.08)	$2.25	$1.00
$1.00	37	8.00%	($2.25)	$3.51	$1.00
$1.00	38	8.00%	($3.51)	$4.87	$1.00
$1.00	39	8.00%	($4.87)	$6.34	$1.00
$1.00	40	8.00%	($6.34)	$7.92	$1.00
$1.00	41	8.00%	($7.92)	$9.64	$1.00
$1.00	42	8.00%	($9.64)	$11.49	$1.00
$1.00	43	8.00%	($11.49)	$13.49	$1.00
$1.00	44	8.00%	($13.49)	$15.65	$1.00
$1.00	45	8.00%	($15.65)	$17.98	$1.00
$1.00	46	8.00%	($17.98)	$20.50	$1.00
$1.00	47	8.00%	($20.50)	$23.21	$1.00
$1.00	48	8.00%	($23.21)	$26.15	$1.00
$1.00	49	8.00%	($26.15)	$29.32	$1.00
$1.00	50	8.00%	($29.32)	$32.75	$1.00
$1.00	51	8.00%	($32.75)	$36.45	$1.00
$1.00	52	8.00%	($36.45)	$40.45	$1.00
$1.00	53	8.00%	($40.45)	$44.76	$1.00
$1.00	54	8.00%	($44.76)	$49.42	$1.00
$1.00	55	8.00%	($49.42)	$54.46	$1.00
$1.00	56	8.00%	($54.46)	$59.89	$1.00
$1.00	57	8.00%	($59.89)	$65.76	$1.00
$1.00	58	8.00%	($65.76)	$72.11	$1.00
$1.00	59	8.00%	($72.11)	$78.95	$1.00
$1.00	60	8.00%	($78.95)	$86.35	$1.00
$1.00	61	8.00%	($86.35)	$94.34	$1.00
$1.00	62	8.00%	($94.34)	$102.97	$1.00
$1.00	63	8.00%	($102.97)	$112.28	$1.00
$1.00	64	8.00%	($112.28)	$122.35	$1.00
($10.06)	65	8.00%	($122.35)	$121.27	($10.06)
($10.06)	66	8.00%	($121.27)	$120.10	($10.06)
($10.06)	67	8.00%	($120.10)	$118.84	($10.06)
($10.06)	68	8.00%	($118.84)	$117.48	($10.06)
($10.06)	69	8.00%	($117.48)	$116.01	($10.06)
($10.06)	70	8.00%	($116.01)	$114.42	($10.06)
($10.06)	71	8.00%	($114.42)	$112.71	($10.06)
($10.06)	72	8.00%	($112.71)	$110.86	($10.06)
($10.06)	73	8.00%	($110.86)	$108.86	($10.06)
($10.06)	74	8.00%	($108.86)	$106.70	($10.06)
($10.06)	75	8.00%	($106.70)	$104.37	($10.06)
($10.06)	76	8.00%	($104.37)	$101.85	($10.06)
($10.06)	77	8.00%	($101.85)	$99.13	($10.06)
($10.06)	78	8.00%	($99.13)	$96.19	($10.06)
($10.06)	79	8.00%	($96.19)	$93.02	($10.06)
($10.06)	80	8.00%	($93.02)	$89.60	($10.06)
($10.06)	81	8.00%	($89.60)	$85.90	($10.06)
($10.06)	82	8.00%	($85.90)	$81.90	($10.06)
($10.06)	83	8.00%	($81.90)	$77.58	($10.06)
($10.06)	84	8.00%	($77.58)	$72.92	($10.06)
($10.06)	85	8.00%	($72.92)	$67.89	($10.06)
($10.06)	86	8.00%	($67.89)	$62.45	($10.06)
($10.06)	87	8.00%	($62.45)	$56.58	($10.06)
($10.06)	88	8.00%	($56.58)	$50.24	($10.06)
($10.06)	89	8.00%	($50.24)	$43.39	($10.06)
($10.06)	90	8.00%	($43.39)	$36.00	($10.06)
($10.06)	91	8.00%	($36.00)	$28.01	($10.06)
($10.06)	92	8.00%	($28.01)	$19.38	($10.06)
($10.06)	93	8.00%	($19.38)	$10.06	($10.06)
($10.06)	94	8.00%	($10.06)	$0.00	($10.06)


The Human Financial Life-Cycle:
Savings, Wealth & Consumption
Wealth Value
Consumption / Spending
Total Wealth
Per Year
Age
$140
$120
$100
$80
$60
$40
$20
$0
$2.00
$0.00
($2.00)
($4.00)
($6.00)
($8.00)
($10.00)
($12.00)
35
39
43
47
51
55
59
63
67
71
75
79
83
87
91

$$\sum_{i=1}^{N+D}(C_i + S_i)\prod_{j=1}^{i}(1+R_j)^{-1}$$

Mean	8.00%
Std.Dev	15.00%

Save / Consume	E.O.Y. Age	Investment Return	DVLP @ 35th Birthday		
			$1.645		
$1.00	**35**	37.13%	$2.256	$1.37	$1.00
$1.00	**36**	17.50%	$1.72	$2.79	$1.00
$1.00	**37**	-13.14%	$0.85	$3.29	$1.00
$1.00	**38**	34.03%	($0.13)	$5.75	$1.00
$1.00	**39**	-14.42%	($1.52)	$5.78	$1.00
$1.00	**40**	34.85%	($2.15)	$9.14	$1.00
$1.00	**41**	5.20%	($4.25)	$10.66	$1.00
$1.00	**42**	-2.83%	($5.53)	$11.33	$1.00
$1.00	**43**	30.54%	($6.34)	$16.10	$1.00
$1.00	**44**	17.76%	($9.58)	$20.14	$1.00
$1.00	**45**	18.60%	($12.46)	$25.07	$1.00
$1.00	**46**	10.63%	($15.97)	$28.84	$1.00
$1.00	**47**	-14.45%	($18.77)	$25.53	$1.00
$1.00	**48**	13.21%	($16.91)	$30.03	$1.00
$1.00	**49**	12.27%	($20.28)	$34.84	$1.00
$1.00	**50**	15.75%	($23.89)	$41.49	$1.00
$1.00	**51**	35.01%	($28.81)	$57.36	$1.00
$1.00	**52**	11.21%	($40.25)	$64.90	$1.00
$1.00	**53**	8.59%	($45.88)	$71.56	$1.00
$1.00	**54**	18.09%	($50.90)	$85.69	$1.00
$1.00	**55**	27.20%	($61.29)	$110.26	$1.00
$1.00	**56**	31.36%	($79.23)	$146.16	$1.00
$1.00	**57**	27.82%	($105.39)	$188.09	$1.00
$1.00	**58**	-24.08%	($135.98)	$143.56	$1.00
$1.00	**59**	3.63%	($104.00)	$149.81	$1.00
$1.00	**60**	-31.61%	($108.82)	$103.14	$1.00
$1.00	**61**	12.76%	($75.10)	$117.42	$1.00
$1.00	**62**	9.19%	($85.81)	$129.30	$1.00
$1.00	**63**	1.54%	($94.78)	$132.31	$1.00
$1.00	**64**	11.67%	($97.26)	$148.87	$1.00
($10.06)	**65**	10.89%	($109.73)	$153.92	($10.06)
($10.06)	**66**	27.90%	($110.52)	$184.00	($10.06)
($10.06)	**67**	30.59%	($128.49)	$227.14	($10.06)
($10.06)	**68**	6.89%	($154.65)	$232.04	($10.06)
($10.06)	**69**	-6.15%	($154.55)	$208.33	($10.06)
($10.06)	**70**	13.17%	($135.61)	$224.37	($10.06)
($10.06)	**71**	38.16%	($142.07)	$296.10	($10.06)
($10.06)	**72**	4.86%	($182.39)	$299.95	($10.06)
($10.06)	**73**	2.18%	($180.71)	$296.20	($10.06)
($10.06)	**74**	25.61%	($174.36)	$359.40	($10.06)
($10.06)	**75**	18.10%	($206.37)	$412.59	($10.06)
($10.06)	**76**	-7.15%	($231.85)	$373.74	($10.06)
($10.06)	**77**	9.10%	($205.93)	$396.78	($10.06)
($10.06)	**78**	-11.84%	($213.69)	$340.92	($10.06)
($10.06)	**79**	-10.72%	($179.51)	$295.40	($10.06)
($10.06)	**80**	-0.18%	($151.29)	$284.82	($10.06)
($10.06)	**81**	-2.26%	($140.97)	$268.53	($10.06)
($10.06)	**82**	-1.82%	($127.94)	$253.77	($10.06)
($10.06)	**83**	-11.09%	($115.74)	$216.68	($10.06)
($10.06)	**84**	-1.84%	($93.96)	$202.82	($10.06)
($10.06)	**85**	-0.21%	($82.35)	$192.35	($10.06)
($10.06)	**86**	2.37%	($72.14)	$186.61	($10.06)
($10.06)	**87**	22.11%	($63.55)	$215.58	($10.06)
($10.06)	**88**	5.53%	($65.31)	$216.87	($10.06)
($10.06)	**89**	-12.93%	($58.30)	$180.06	($10.06)
($10.06)	**90**	17.81%	($42.00)	$200.27	($10.06)
($10.06)	**91**	4.63%	($37.62)	$199.02	($10.06)
($10.06)	**92**	5.38%	($28.83)	$199.13	($10.06)
($10.06)	**93**	3.54%	($19.78)	$195.76	($10.06)
($10.06)	**94**	14.54%	($10.06)	$212.69	($10.06)


The Human Financial Life-Cycle:
Savings, Wealth & Consumption
Wealth Value
Consumption / Spending
Total Wealth
Per Year
Age
$450
$400
$350
$300
$250
$200
$150
$100
$50
$0
$2.00
$0.00
($2.00)
($4.00)
($6.00)
($8.00)
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