FINE6860: Lecture #3
Basic Models of Interest

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Summer 2006

1 Continuously Compounded Interest

Our models will mostly be developed in continuous time. This means that money grows as a result of the force of interest in a continuous manner. To maintain consistency, I will use the letter r to denote the current continuously compounded (CC) rate of interest. The relationship between the nominal CC rate r and the effective annual rate $\exp\{r\} - 1$, is via the exponential operator, or its inverse the natural logarithm. For example, if the effective annual rate is 10%, the continuously compounded rate will be (a lower) $\ln[1 + 0.10] = 9.531\%$ per annum.

	$1...Based on Compounding Frequency	
Rate	Annual $(n=1)$	Quarterly $(n=4)$
4%	$\left(1+\frac{0.04}{1}\right)^1 = 1.04$	$\left(1+\frac{0.04}{4}\right)^4 = 1.0406$
5%	$\left(1+\frac{0.05}{1}\right)^1 = 1.05$	$\left(1+\frac{0.05}{4}\right)^4 = 1.05095$
6%	$\left(1+\frac{0.06}{1}\right)^1 = 1.06$	$\left(1+\frac{0.06}{4}\right)^4 = 1.06136$
7%	$\left(1+\frac{0.07}{1}\right)^1 = 1.07$	$\left(1+\frac{0.07}{4}\right)^4 = 1.07186$
8%	$\left(1+\frac{0.08}{1}\right)^1 = 1.08$	$\left(1+\frac{0.08}{4}\right)^4 = 1.08243$
10%	$\left(1+\frac{0.10}{1}\right)^1 = 1.10$	$\left(1+\frac{0.10}{4}\right)^4 = 1.10381$
12%	$\left(1+\frac{0.12}{1}\right)^1 = 1.12$	$\left(1+\frac{0.12}{4}\right)^4 = 1.12551$

The more frequently we compound interest, the greater the sum of money available at the end of the year.

	$1...Based on Compounding Frequency	
Rate	Daily $(n = 365)$	Continuous $(n = \infty)$
4%	$\left(1 + \frac{0.04}{365}\right)^{365} = 1.040\,81$	$e^{0.04\times 1} = 1.04081$
5%	$\left(1 + \frac{0.05}{365}\right)^{365} = 1.051\,27$	$e^{0.05\times 1} = 1.05127$
6%	$\left(1 + \frac{0.06}{365}\right)^{365} = 1.061\,83$	$e^{0.06\times 1} = 1.06184$
7%	$\left(1 + \frac{0.07}{365}\right)^{365} = 1.072\,5$	$e^{0.07\times 1} = 1.07251$
8%	$\left(1 + \frac{0.08}{365}\right)^{365} = 1.083\,28$	$e^{0.08\times 1} = 1.08329$
10%	$\left(1 + \frac{0.10}{365}\right)^{365} = 1.105\,16$	$e^{0.10\times 1} = 1.10517$
12%	$\left(1 + \frac{0.12}{365}\right)^{365} = 1.127\,47$	$e^{0.12\times 1} = 1.12750$

Notice that going from a 12% rate compounded annually (i.e. simply interest) to a 12% rate compounded continuously, the additional gain is on the order of 75 basis points.

Mathematically we are building on the relationship:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r. \tag{1}$$

This can be formally proved by defining a new variable:

$$y := \left(1+\frac{r}{n}\right)^n \tag{2}$$

and then talking natural logarithms of both sides so that:

$$\ln[y] = n\ln[1+\frac{r}{n}]. \tag{3}$$

This leads to:

$$\lim_{n\to\infty} \ln[y] = \lim_{n\to\infty} \frac{\ln[1+\frac{r}{n}]}{1/n}. \tag{4}$$

We now invoke LeHospital's rule. If we take derivatives of the numerator and denominator we are left with:

$$\lim_{n\to\infty} \frac{\ln[1+\frac{r}{n}]}{1/n} = \lim_{n\to\infty} \frac{r}{1+\frac{r}{n}} = r, \tag{5}$$

which then leads to

$$\lim_{n\to\infty} \ln[y] = r \tag{6}$$

and therefore

$$\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^n = e^r. \tag{7}$$

2 Discount Factors:

Using our terminology, the discounted value of one dollar to be received at time T is denoted and equal to:

$$d(t) = e^{-rt}. \tag{8}$$

The symbol $d(t)$ will often be referred to as a discount factor and can be envisioned as an exchange rate between one dollar today and one dollar at time T. With a discount factor (function) in our hands we don't have to worry about the precise interest rate r and we can compute the present value of *any* cash flow C, by simply multiplying it by $d(T)$.

For example, when $r = 5\%$ and $T = 10$ we obtain a discount factor of $d_{10} = e^{-0.05\times 10} = 0.606\,5$, but when $r = 3\%$ and $T = 10$ years, the discount factor is a higher $d_{10} = e^{-0.03\times 10} = 0.7408$. Stated differently, a dollar in ten years is worth 60.65 cents today when the interest rate is 5% and it is worth 75.08 cents today when the interest rate is 3%.

	$1 waiting for returns...in years	
Rate	Double ($C = \$2$)	Triple ($C = \$3$)
4%	$\frac{1}{0.04}\ln[2] = 17.3286$	$\frac{1}{0.04}\ln[3] = 27.4653$
5%	$\frac{1}{0.05}\ln[2] = 13.8629$	$\frac{1}{0.05}\ln[3] = 21.9722$
6%	$\frac{1}{0.06}\ln[2] = 11.5524$	$\frac{1}{0.06}\ln[3] = 18.3102$
7%	$\frac{1}{0.07}\ln[2] = 9.9021$	$\frac{1}{0.07}\ln[3] = 15.6944$
8%	$\frac{1}{0.08}\ln[2] = 8.6643$	$\frac{1}{0.08}\ln[3] = 13.7326$
10%	$\frac{1}{0.10}\ln[2] = 6.9314$	$\frac{1}{0.10}\ln[3] = 10.9861$
12%	$\frac{1}{0.12}\ln[2] = 5.7762$	$\frac{1}{0.12}\ln[3] = 9.1551$

Remember that there is an inverse relationship between the interest rate and the discount factor. If $r \uparrow$ then $d(t) \downarrow$. Also, when $t \uparrow$ then $d(t) \downarrow$ as well.

Question: How long does one dollar have to be invested before it doubles, triples and quadruples in value assuming it is invested at a rate of r (cc) per annum?

Answer: If we are interested in \$1 growing into \$$C$, we must solve:

$$e^{rT} = C \qquad \Longleftrightarrow \qquad T = \frac{1}{r} \ln [C] \tag{9}$$

Remember that using continuous compounding, the variable T is expressed in decimal form, which means that if we want to obtain its value in months, we must calculate $12T$, and for weeks we must calculate $52T$, etc. Notice again the inverse relationship between the interest rate r and the time needed to grow to a fixed dollar sum of C.

Thus, at a 12% interest rate you have to wait 5.77 years for your money to double and at 4% the wait is 17.32 years.

3 How Accurate is the Rule of 72?

Practitioners often invoke something called the "rule of 72" which claims that if you divide 72 into the *effective* annual interest rate, you will get an estimate for the number of years you must wait for your money to double. To compare this popular rule to the results in the table, I first must convert the 4% and 12% which are continuously compounded rates into $e^{0.04} - 1 = 0.0408$ and $e^{0.12} - 1 = 0.1274$ which are effective.

Using the rule of 72, we get $72/4.08 = 17.647$ which is a bit higher than the correct 17.32 years and $72/12.74 = 5.65$ which is slightly lower than the correct 5.77 years.

Note that if we use the "rule of 72" with a denominator that is continuously compounded instead of effective, the error in this approximation can easily be written as:

$$\text{Error in Rule of 72} := \frac{1}{r}\left[\frac{72}{100} - \ln[2]\right] = \frac{0.02685}{r}.$$

When the valuation rate is larger than 2.68% the "approximatin bias" is less than one year and when the valuation rate is smaller than 2.68% the "approximation bias" is greater than one year. Under this implementation the error declines in r, and is always positive, which means that the rule overestimates the waiting time.

Interestingly enough, since $100\ln[2] \approx 69.3$, a more accurate rule would have been the "rule of 69" when the interest rate is continuously compounded. In this case the error between the correct result and approximation would be smaller.

4 A Term Structure of Interest Rates

In practice the valuation rate can depend on the maturity time t. Therefore, when using a time-dependent interest rate I will adhere to the notation $r(t)$ to remind the reader that the interest rate curve depends on the maturity. In this case the discount factor would retain the same functional form:

$$d(t) = e^{-r(t)t}, \tag{10}$$

with the understanding the omitting the subscript on the r, will imply a constant or flat valuation curve.

For example, assume that the time-dependent continuously compounded interest rate $r(t)$ is equal to:

$$r(t) = a - \frac{b}{t+1}, \quad t \geq 0. \tag{11}$$

In this case, when $a = 5\%$ and $b = 2\%$ we have $r(10) = 0.04818$ and when $a = 6\%$ and $b = 2\%$ we have $r(10) = 0.05818$. Note that in this model as $T \to \infty$ the interest rate converges to a from below since the second portion converges to zero.

In this case the discount factor d_t would be:

$$d(t) = e^{-at+bt/(t+1)}.$$

Think about what this "term structure of interest rate" looks like graphically. Let us plot in Excel under various values of a and b.

5 Zeros and Coupon Bonds

Imagine a bond which matures in T years and pays annual coupons of c times face value. Assume these coupons are paid each and every day, in the amount of $c/365$. What is this bond "worth" today when interest rates in the market are at r?

Well, if we assume these coupons are paid in *continuous-time* instead of daily – and this won't make a big difference as we saw earlier – then a model value of this bond can be written as:

$$V(c, r, T) = \int_0^T ce^{-rs} ds + e^{-rT}. \tag{12}$$

Thus, a $\$10,000$ face value bond – which pays annual coupons of $10,000c$ – would have a model value of $100,00V(c, r, T)$. A $\$100,000$ face value bond – which pays annual coupons of $100,000c$ – would have a model value of $100,000V(c, r, T)$.

In either event, after some simple calculus applied to the valuation equation, we obtain that the model value of a "generic" bond can be written as:

$$V(c,r,T) = \frac{c}{r}\left(1 - e^{-rT}\right) + e^{-rT}. \qquad (13)$$

Note that if $c = r$, we get $V(r,r,T) = 1$. In words, when the valuation rate is precisely equal to the coupon yield on the bond, it will have a "par" (a.k.a. equal to face) model value. When $c > r$, the bond will have a model value of $V(c,r,T) > 1$, and when $c < r$ the value of the bond will be $V(c,r,T) < 1$.

The following tables provide numbers for "generic" bond values as a function of the valuation rate r, the coupon yield c and the maturity T.

5% Coupon Yield (c)	What is the Bond Value?
Valuation Rate (r)	$T = 5$ Years Maturity
4%	$V(0.05, 0.04, 5) = 1.0453$
5%	$V(0.05, 0.05, 5) = 1.0$
6%	$V(0.05, 0.06, 5) = 0.9568$

When we increase the maturity from $T = 5$ years to $T = 10$ years notice the impact on the bond value. Remember also that these numbers are a fraction of "face value." So, for example, a five-year 6% bond is worth $9,568 when the face value is $10,000 and the valuation rate is $r = 6\%$.

5% Coupon Yield (c)	What is the Bond Value?
Valuation Rate	$T = 10$ Years Maturity
4%	$V(0.05, 0.04, 10) = 1.0824$
5%	$V(0.05, 0.05, 10) = 1.0$
6%	$V(0.05, 0.0, 10) = 0.9248$

6 Arbitrage: Linking Value and Price

Note that I am careful to distinguish between the "model value" of the generic bond which is based on formulae and assumption versus the "market price" of the bond which is an actual number at which investors can buy and sell the bond. In many cases the "model value" of a financial instrument can differ from the "market price" of the same instrument and we will discuss these reasons at length, later in the analysis.

However, when the valuation rate is r and an investor can borrow as well as lend money at this valuation rate, then the above-mentioned "model value" $V(c, r, T)$ must also be the "market price" of the bond. If not, there is an arbitrage or opportunity for risk-less profit. This imbalance can not persist for very long – would you leave a $100 dollar bill on the floor? – and eventually the market price will converge towards the model value.

7 Bonds: Non-Flat Term Structure

If the continuously compounded valuation rate $r(t)$ is a function of time, the the fundamental bond valuation equation (13) must be written as:

$$V(c, r(t), T) = c \int_0^T e^{-r(s)s} ds + e^{-r(T)T} \quad (14)$$

For example, if we assume:

$$r(t) = a - b\left(\frac{1}{t+1}\right), \quad (15)$$

then:

$$V(c, r(t), T) = c \int_0^T e^{-(a-b/(s+1))s} ds + e^{-(a-b/(T+1))T}. \quad (16)$$

Figure #1 displays the valuation rate $r(t)$ over 30 years for various choices of $\{a, b\}$ in the above equation. Note the impact of changing b on the "big picture".

8 Bonds: Non-Constant Coupons

If the coupon yield $c(t)$ is also a function of time, then the fundamental bond valuation equation (13) must be written as:

$$V(c(t), r(t), T) = \int_0^T c(s)e^{-r(s)s}ds + e^{-r(T)T} \qquad (17)$$

For example, if a 20-year bond with a face (or principal) value of \$10,000 pays an annual coupon of \$1,000 that declines by 7% each year, then under a constant valuation rate $r = 10\%$, the model bond value would be expressed as:

$$\begin{aligned} V &= \int_0^{20} 1000e^{-(0.07)s}e^{-(0.10)s}ds + 10000e^{-(0.10)(20)} \\ &= \$7,039.39 \end{aligned} \qquad (18)$$

The first term in the integrand captures the declining coupon and the second term is the present value factor which brings all the coupons back to time zero.

More generally, a bond with a face value of F that pays a coupon of cF that declines by λ% each year, would have a value of:

$$\begin{aligned} V &= cF\int_0^T e^{-(r+\lambda)s}ds + Fe^{-rT} \\ &= \frac{cF}{r+\lambda}\left(1 - e^{-(r+\lambda)T}\right) + Fe^{-rT} \end{aligned} \qquad (19)$$

Note that when the bond becomes a perpetuity, which means that $T \to \infty$, the bond value will converge to a simple $V = cF/(r+\lambda)$.

It might seem artificial and unrealistic to have a bond that pays coupons in this way, but later we shall see a number of applications of this concept.

9 Taylor's Approximation

In this section I investigate the sensitivity or impact of the valuation rate r, on the "generic" bond equation $V(c, t, T)$. Specifically, I am interested in "how much" the bond value will change when we increase or decrease the rate r by a small amount Δr.

Economic intuition dictates that when $\Delta r > 0$ the change in the value of the bond will be negative and when $\Delta r < 0$ the change in the value of the bond will be positive. **Why?**

If you remember your calculus, we can approximate the change in the value of any continuous function by taking derivatives of the given function and applying Taylor's theorem. According to Taylor approximation:

$$\begin{aligned} & V(c, r + \Delta r, T) - V(c, r, T) \\ \approx \; & (\Delta r) V'(c, r, T) + \frac{(\Delta r)^2}{2} V''(c, r, T), \end{aligned} \tag{20}$$

where $V'(c, r, T)$ and $V''(c, r, T)$ denote the first and second derivative of the bond equation relative to the valuation rate r. The intuition for this relationship is that a "small" change in the rate r, will trigger a "small" change in the bond, where the relationship between these two changes is determined by how quickly or rapid the bond function $V(c, r, T)$ moves when plotted against r.

It is convenient to re-write equation (20) by dividing both sides by the bond value $V(c, r, T)$, which leads to:

$$\frac{V(c, r + \Delta r, T) - V(c, r, T)}{V(c, r, T)}$$
$$\approx (\Delta r)\frac{V'(c, r, T)}{V(c, r, T)} + \frac{(\Delta r)^2}{2}\frac{V''(c, r, T)}{V(c, r, T)}. \quad (21)$$

In English, the "relative change" in the bond value as a result of a movement in the rate r, can be approximated by the sum of two quantities on the right-hand side of equation (21). Finally, given the centrality of this approximation in a number of places thru-out the material, I will use the symbol $D(c, r, T)$ to denote:

$$D(c, r, T) = -\frac{\partial V(c, r, T)/\partial r}{V(c, r, T)} \quad (22)$$

and the symbol $K(c, r, T)$ to denote:

$$K(c, r, T) = \frac{\partial^2 V(c, r, T) \partial r^2}{V(c, r, T)} \quad (23)$$

where in both definitions the derivative with respect to the rate r is now stated explicitly.

Later I will explain why I have decided to define $D(c, r, T)$ as "negative" the derivative, which might seem odd at first glance. Nevertheless, these shortcuts lead us from equation (21) to the abbreviated approximation:

$$\text{\% Change in Bond Value} \;\approx\; -(\Delta r)D + \frac{(\Delta r)^2}{2}K. \qquad (24)$$

Note, also, that nowhere in this approximation do we explicitly use the functional form of the bond value itself. Indeed, even if the pricing equation is some complicated function of valuation rates, coupon yields and time horizons, the relationship in equation (#) should hold.

10 Explicit Values for D and K

Recall that in the "generic" case, the explicit definition of the bond value was:

$$V(c,r,T) = \frac{c}{r}\left(1 - e^{-rT}\right) + e^{-rT} \tag{25}$$

which came from integrating the coupon rate c against the discount function e^{-rs} and then adding the discounted value of the face. Using this expression we can obtain explicit values for D and K by taking the appropriate partial derivatives in equation (22) and (23). This leads to:

$$D(c,r,T) = -\frac{c(e^{-rT} - 1 + rTe^{-rT}) - r^2Te^{-Tr}}{cr(1 - e^{-Tr}) + r^2e^{-Tr}} \tag{26}$$

and

$$K(c,r,T) = \frac{c(2 - e^{-rT}(2 + 2rT + r^2T^2)) + r^3T^2e^{-rT}}{cr^2(1 - e^{-Tr}) + r^3e^{-rT}}. \tag{27}$$

Despite the messy-looking expressions for both D and K, a number of important insights can be obtained from "staring" at the equations long enough.

Figure #2 provides some graphical intution for the relationship between V, K and D as a function of c, r and T.

First, with regards to $D(c, r, T)$ – which can be interpreted as the bond value's derivative scaled by the bond value's price – notice that when the valuation rate r is equal to the coupon yield c, the expression simplifies to:

$$D(c, c, T) = \frac{1 - e^{-cT}}{c}, \tag{28}$$

which converges to T when $c \to 0$.

Along the same lines, note that when $c = 0$ (which, recall, is a zero coupon bond) the value of D simplifies to:

$$D(0, r, T) = T,$$

independently of r, which happens to be the exact maturity of the zero-coupon bond. This is why it is common to measure D in units of years. Later I will derive a deeper connection between D and actual units of time.

Moving on to K – the bond value's second derivative scaled by the bond values price – notice that when the coupon yield c is equal to the valuation rate r, we get a much simpler:

$$K(c, c, T) = \frac{2(1 - Tce^{-cT} - e^{-cT})}{c^2}. \tag{29}$$

Furthermore, when $c = 0$ and the bond is of the zero-coupon variety, the value collapses to:

$$K(0, r, T) = T^2, \tag{30}$$

hence it is common to measure K in units of years squared.

11 Numerical Examples of D & K

Lets start with two bonds. Bond #1 has a face value of $10,000 paying a continuous coupon yield of $c = 11\%$ and maturing in $T = 17.20$ years. The current (valuation) rate in the market is assumed to be $r = 7\%$ and the bond value is therefore:

$$\$10,000V(0.11, 0.07, 17.2) \approx \$14,000. \quad (31)$$

At the same time, another $10,000 face-value bond #2 paying a coupon of $c = 10\%$ and maturity in $T = 38.69$ years is also "worth" $\$14,000$ under the current $r = 7\%$ valuation rate since:

$$\$10,000V(0.10, 0.07, 38.69) \approx \$14,000 \quad (32)$$

The D and K values of the two bonds are as follows. For bond #1, equation (26) leads to $D(0.11, 0.07, 17.2) = 9.1185$ years and equation (27) leads to $K(0.11, 0.07, 17.2) = 119.002$ units, respectively. Note that the D value is much lower than the maturity of $T = 17.2$ years and the K value is much lower than $T^2 = 295.84$ years-squared.

The D and K values of bond #2 are $D(0.10, 0.07, 38.69) = 12.8162$ years and $K(0.10, 0.07, 38.69) = 283.010$ units respectively. Of course, the larger values come from the longer maturity of bond #2. Interestingly, the 20 years maturity of bond #2 adds less than four years to the D value. In other words, the sensitivity of the bond value to changes in the rate r is not that much greater for bond #2 compared to bond #1.

Now let us get back to our approximation. Both bonds are "worth" $14,000. Assume the valuation (or market) interest rate $r = 7\%$ changes from $r = 7\%$ to $7\% + \Delta r$ over a (very) short period of time so that the maturity of the two bonds are still 17.2 years and 28.69 years.

Using the Taylor's $D\&K$ method – as presented in equation (24) – the change in the value (or price) of the bond will be approximated by:

$$V(c,r,T)\left(-(\Delta r)D+\frac{(\Delta r)^2}{2}K\right). \tag{33}$$

For example, when $\Delta r = 0.01$, which is a one percent (or 100 basis point) increase in the valuation rate, we get Bond #1:

$$\begin{aligned}&\approx 14000+14000(-(0.01)(9.\,1185)+\frac{(0.01)^2}{2}119.\,002)\\&=\$12,806.71,\end{aligned} \tag{34}$$

and for bond #2 we get,

$$\begin{aligned}&\approx 14000+14000(-(0.01)(12.\,8162)+\frac{(0.01)^2}{2}283.\,010)\\&=\$12,403.84\end{aligned} \tag{35}$$

The value of both bonds will fall when interest rates increase, but the impact of this change on bond #2 will be greater than the impact on bond #1. In fact, bond #2 will drop in value by $400 more as a result of the greater sensitivity D to changes in rates.

Note that if we use the precise "genetic bond" formula for the value of both under the new interest rate $r = 8\%$, we get:

$$10000V(0.11, 0.08, 17.2) = \$12,802.80 \tag{36}$$

and

$$10000V(0.10, 0.08, 38.69) = \$12,386.84, \tag{37}$$

respectively. The message is clear. Taylor's $D\&K$ approximation gives us numbers that are within a few dollars of the true bond value. The following table provides a more extensive example of how good (or not) the approximation "works" when we compare to the correct bond value. In the second column I have computed the new bond value using only the 1st derivative K in Taylor's approximation and in the third column I have used both the first and second derivative.

$c = 7\%$	How Good is the Approximation?		
$r = 7\%$	Bond is Worth $10,000....and then rates change:		
$T = 30$	1st Derivative	+ 2nd Derivative	$10000V(c, r, T)$
Δr	D Only	D & K	Exact Value
+2.5%	$\$6,865.92$	$\$7,657.22$	$\$7,520.64$
+1%	$\$8,746.37$	$\$8,872.98$	$\$8,863.40$
+0.5%	$\$9373.18$	$\$9,404.83$	$\$9,403.60$
+0.1%	$\$9,874.64$	$\$9,875.90$	$\$9,875.89$
0%	$\$10,000.00$	$\$10,000.00$	$\$10,000.00$
-0.1%	$\$10,125.36$	$\$10,126.63$	$\$10,126.64$
-0.5%	$\$10,626.82$	$\$10,658.47$	$\$10,659.79$
-1.0%	$\$11,253.63$	$\$11,380.24$	$\$11391.17$
-2.5%	$\$13,134.08$	$\$13,925.39$	$\$14,115.33$

12 Introduction to Duration and Convexity

Let us go back to first principles and carefully examine the definition of Taylor's D, using the integral representation of the "generic" bond value:

$$D(c,r,T) = \frac{-\frac{\partial}{\partial r}\left(\int_0^T ce^{-rs}ds + e^{-rT}\right)}{\int_0^T ce^{-rs}ds + e^{-rT}}. \tag{38}$$

The numerator is (minus) the first derivative of the bond price with respect to the valuation rate and the denominator is the bond value itself. Remember that the derivative "operator" can be moved inside the integral and then "used" on the integrand so that the entire $D(c,r,T)$ can be re-written as:

$$D(c,r,T) = \int_0^T s\left(\frac{ce^{-rs}}{V(c,r,T)}\right)ds + T\left(\frac{e^{-rT}}{V(c,r,T)}\right). \tag{39}$$

Stare at this expression for a while. We see that the $D(c,r,T)$ function can <u>also</u> be identified as a type of weighted average. *The duration of the bond value is the weighted average of the time-to-payment where the weights are the share of the bond's cash flow in present value terms.*

Thus, from here on we lavel $D(c,r,T)$ the bond's duration...and $K(c,r,T)$ is called the convexity of the bond.