FINE6860: Lecture #5
Models for Pension Annuities

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1 Pension Annuity Prices

Table #1 displays the actual prices (quotes) of pension annuities for individuals at various ages. These quotes are based on a $100,000 "premium" or deposit that is paid at the time of purchase with funds from a tax-sheltered savings plan. I have displayed the payouts based on the average of the "best" companies quoting in the U.S. in early January 2005.

$100,000 Premium Single-Life Pension Annuity								
Monthly Income Starts Immediately After Purchase								
Period	Age 50		Age 65		Age 70		Age 80	
Certain	Ma	Fe	Ma	Fe	Ma	Fe	Ma	Fe
0 yr	$514	$492	$655	$605	$747	$677	$1073	$961
10 yr	$509	$490	$630	$592	$694	$649	$841	$812
20 yr	$498	$484	$569	$555	$591	$583	$585	$585
Table #1: Source CANNEX Jan 2005								

Joint-life pension annuities are displayed in Table #2.

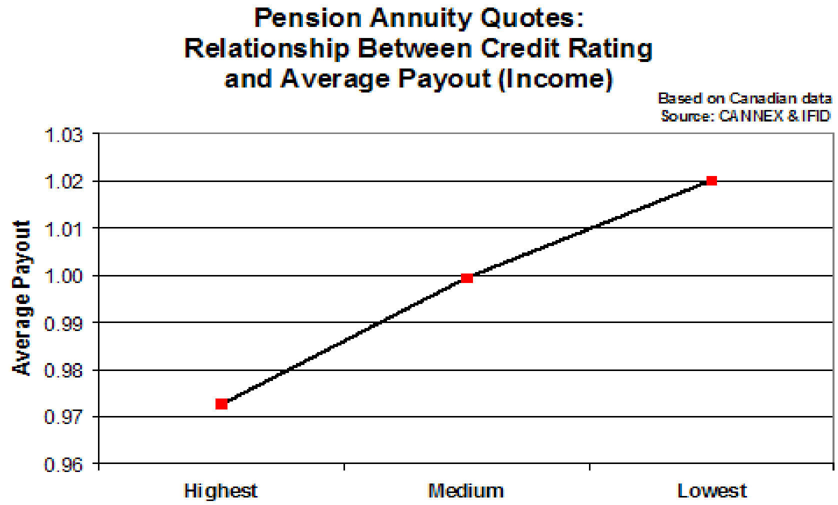
$100,000 Premium Joint-Life Pension Annuity				
Monthly Income Starts Immediately After Purchase				
Period	Age of Male and Female			
Certain	50 yrs	65 yrs	70 yrs	80 yrs
0 yr	$465	$545	$597	$791
10 yr	$465	$544	$594	$739
20 yr	$465	$533	$547	$601
Table #2: Source CANNEX Jan 2005				

2 What Does the Company do with the Premium?

Table #3 compares "actual" bond yields around the same time.

A Quick Comparison with the Bond Market			
Coupon Yield	Maturity	Price of Bond	Yield to Maturity
3 + 1/8 %	≈ 2 yrs	99.78	3.24
3 + 5/8 %	≈ 5 yrs	99.72	3.68
4 + 1/4 %	≈10 yrs	100.88	4.14
7 + 1/2 %	≈ 20 yrs	136.69	4.64
5 + 3/8 %	≈ 30 yrs	111.56	4.61

Table #3: Source *Wall Street Journal* (January 2005)



1.The Credit Effect

Figure #1 provides some further evidence for the credit rating "hypothesis".

3 Valuation of Pension Annuities: General

The insurance company commits to pay the annuitant $\$1$ dollars per year for the rest of the annuitant's life. Assuming an effective valuation rate of R per annum, the **Discounted Value of the Pension Annuity (DVPA)** is:

$$\text{DVPA} = \sum_{i=1}^{D} \frac{1}{(1+R)^i}, \tag{1}$$

where D is the random (integer) number of years until death. The integral version of this expression, when payments are made in continuous-time, is:

$$\text{DVPA } = \int_0^{\mathbf{T}_x} e^{-rt} dt = \int_0^{\infty} e^{-rt} 1_{\{\mathbf{T}_x \geq t\}} dt, \tag{2}$$

where $\mathbf{T}_x$ is the remaining lifetime random variable defined earlier and the "indicator function" $1_{\{\mathbf{T}_x \geq t\}}$ takes on the value of one when $\mathbf{T}_x \geq t$ and zero when $\mathbf{T}_x < t$. I can not over-emphasize enough that DVPA is a random variable.

The expected value of this random variable is called the **Immediate Pension Annuity Factor (IPAF):**

$$\begin{aligned}\overline{a}_x &= E\left[\int_0^{\mathbf{T}_x} e^{-rt}dt\right] = \int_0^{\infty} e^{-rt}({}_tp_x)dt \\ &= \int_0^{\infty} e^{-\left(rt+\int_0^t \lambda(x+s)ds\right)}dt \qquad (3)\end{aligned}$$

where the new symbol a_x – which I will be using repeatedly – is pronounced "ay bar ex". We emphasize the word "immediate" to remind the reader that income is assumed to begin immediately upon purchase. The expectation in equation (3) can be converted to a survival probability curve since $E[1_{\{\mathbf{T}_x \geq t\}}] = ({}_tp_x)$. The second equality comes from the definition of the survival probability.

4 Valuation of Pension Annuities: Exponential

When $\mathbf{T}_x$ is exponentially distributed, which implies that $({}_tp_x) = e^{-\lambda t}$, the annuity factor from equation (3) collapses to:

$$\overline{a}_x = \int_0^\infty e^{-(r+\lambda)dt}dt = \frac{1}{r+\lambda}. \tag{4}$$

For example, when $r = 5\%$ and $\lambda = 5\%$ the annuity factor is $1/(0.05 + 0.05) = \$10.0$ per dollar of lifetime income. If $r = 4\%$ and $\lambda = 6\%$ the annuity factor is (still) \$10 and the same is true when $\lambda = 6\%$ and $r = 4\%$. Note how it is only the sum of r and λ that matters, and not the individual components. The interest rate r and the instantaneous force of mortality (IFM) λ have the exact same impact on the annuity factor. They both discount the future to the present. One adjusts for the value of money and the other adjust for the value of mortality. And, despite the fact that equation (4) only "works" under exponential mortality, the tight connection between r and the general $\lambda(x)$ curve will appear again many times.

5 The Wrong Way to Value Pension Annuities

A common mistake is to value pension annuities by arguing that income will be received "on average" until the expected remaining lifetime (ERL), which using our notation is $E[\mathbf{T}_x]$. To understand why this is wrong (or at best, a biased approximation) think of the remaining lifetime random variable under an exponential distribution. In this case the discounted value of income until the ERL is:

$$\int_0^{1/\lambda} e^{-rt}dt = \frac{e^{-r/\lambda}}{-r} + \frac{1}{r} = \frac{1}{r}(1 - e^{-r/\lambda}). \tag{5}$$

And, since we can approximate the exponential term by $e^{-r/\lambda} \approx 1 - r/\lambda$,this leaves us with an approximate integral value of $1/\lambda$, which is larger than the "correct" annuity factor $1/(r+\lambda)$. For example, when $r = 5\%$ and $\lambda = 4\%$, which leads to an expected remaining lifetime of 25 years, the correct annuity factor as per equation (4) is $1/(0.09) = \$11.111$ per dollar of lifetime income. However, under the incorrect formula listed above, the annuity factor would be \$14.27 which is higher by more than \$3 per dollar of lifetime income. Stated differently, a fixed premium of \$100,000 converted into a pension annuity *should* provide an annual income $\$100,000/11.11 = \$9,000$ under exponential mortality and not $\$100,000/14.27 \approx \$7000$. Using the erroneous method will lead to less income. This error is present regardless of the particular law of mortality that is used for valuation purposes.

On a slightly more technical level, we conclude this particular discussion by stating that:

$$\int_0^{E[\mathbf{T}_x]} e^{-rt}dt > E\left[\int_0^{\mathbf{T}_x} e^{-rt}dt\right], \tag{6}$$

which is a general way of arguing that the "incorrect" annuity factor on the left-hand side is always greater than the "correct" annuity factor on the right-hand side. This fact is also a corollary of *Jensen's inequality* in the mathematical literature.

6 Valuation of Pension Annuities: Gompertz-Makeham

Recall that under the GoMa law of mortality, the IFM obeys the relationship:

$$\lambda(x) = \lambda + \frac{1}{b}e^{\left(\frac{x-m}{b}\right)}. \tag{7}$$

The survival probability was shown to be:

$$({}_tp_x) = \exp\left\{-\lambda t + b(\lambda(x) - \lambda)(1 - e^{t/b})\right\}. \tag{8}$$

Consequently – based on the formula in equation (3) – the annuity factor under GoMa can be expressed as:

$$\begin{aligned} \overline{a}_x &= \exp\{b(\lambda(x) - \lambda)\} \\ &\times \int_0^\infty \exp\left\{-(\lambda + r)t - b(\lambda(x) - \lambda)e^{t/b}\right\} dt, \end{aligned} \tag{9}$$

We now substitute the change-of-variable $s = \exp\{t/b\}$, and $ds = dt \exp\{t/b\}/b$, so that $ds/s = dt/b$, and $s^b = \exp\{t\}$, which leaves us with:

$$\begin{aligned} \overline{a}_x &= b \exp\{b(\lambda(x) - \lambda)\} \\ &\times \int_1^\infty s^{-(\lambda+r)b-1} \exp\{-b(\lambda(x) - \lambda)s\} ds. \end{aligned} \tag{10}$$

Finally, we substitute a second change-of-variable and let

$w = b(\lambda(x) - \lambda)s$, so that $dw = b(\lambda(x) - \lambda)ds$, and therefore:

$$\begin{aligned}\overline{a}_x &= b(b\lambda(x) - b\lambda)^{(\lambda+r)b} \\ &\quad \times \exp\left\{b(\lambda(x) - \lambda)\right\}\Gamma\left(-(\lambda + r)b, b(\lambda(x) - \lambda)\right). \quad (11)\end{aligned}$$

Recall that the Greek symbol $\Gamma(\cdot, \cdot)$ – capital Gamma – is shorthand for the Incomplete Gamma (IG) function defined as:

$$\Gamma(u, v) = \int_v^\infty e^{-t} t^{(u-1)} dt. \quad (12)$$

The IG function can be written in terms of the Cumulative Distribution Function (CDF) of the Gamma random variable $\mathbf{G}$ together with the standard Gamma function $\Gamma(u)$, via the relationship:

$$\Gamma(u, v) = \Gamma(u)(1 - \text{GammaDist}(v; u, 1)),$$

where $\text{GammaDist}(v; u, 1) = \Pr[\mathbf{G} \leq v]$ is the CDF evaluated at v, under parameters $(u, 1)$. For example the value of $\Gamma(2, 3) \approx 0.199$ and $\Gamma(3, 2) \approx 1.353$, with three digits of precision. This numbers can also be obtained by computing $\Gamma(2) = 1.0$ and multiplying by $(1 - \text{GammaDist}\,(3; 2, 1)) = 0.199$, to recover the first expression $\Gamma(2, 3)$.

In the end this leads to the final (and main) expression:

$$\overline{a}_x = \frac{b\Gamma\left(-(\lambda+r)b, \exp\{\frac{x-m}{b}\}\right)}{\exp\left\{(m-x)(\lambda+r) - \exp\{\frac{x-m}{b}\}\right\}}, \tag{13}$$

where the last part of the "story" is obtained by recognizing that $(b\lambda(x) - b\lambda)^{(\lambda+r)b}$ can be simplified to $\exp\{(x-m)(\lambda+r)\}$, using the original definition of the IFM $\lambda(x)$ in equation (7).

This might all appear overwhelming at first, so here are a number of numerical examples to help with some intuition for the formulas. Assume that $\lambda = 0$, $m = 86.34$ and $b = 9.5$ for the GoMa law – which were the best fitting parameters to the unisex RP2000 mortality table analyzed in an earlier chapter – in all of the following situations. Under a valuation rate of $r = 4\%$ the "annuity factor" for ages $x = 65, 75$ and 85 are $a_{65} = 12.454$, $a_{75} = 8.718$ and $a_{85} = 5.234$ respectively. The intuition should be clear. The older the annuitant at the point of "annuitization" the lower is the value of each dollar of lifetime income. These numbers can obviously be scaled up. A pension annuity that pays $650 per month – which is $7,800 per year – has a value of $(12.454)(7800) = \$97,141$ at age 65. This number is not far from the $100,000 premium (price, cost) listed in Table #1, which entitled a 65-year-old male annuitant to $655 for life. The reason it is not *exactly* equal is likely due to different interest rates, mortality estimates and commissions embedded within the quoted annuity price. We will return to this issue later.

If we increase the GoMa λ value from $\lambda = 0$ to $\lambda = 0.01$ –

while maintaining the same m, b and $r = 4\%$ value – the annuity factors are reduced to $a_{65} = 11.394$, $a_{75} = 8.181$ and $a_{85} = 5.026$ respectively. The actuarial reason for this is that a positive λ parameter increases the instantaneous force of mortality and "kills" more people. This means the insurance company has to pay less, which reduces the annuity factor at all annuitization ages.

Buying Lifetime Income of $1			
Immediate Pension Annuity Factor (IPAF) $\overline{a}_x$			
Starting at:	$r = 4\%$	$r = 6\%$	$r = 8\%$
Age 55	$\$15.822$	$\$12.700$	$\$10.480$
Age 65	$\$12.454$	$\$10.474$	$\$8.963$
Age 75	$\$8.718$	$\$7.696$	$\$6.857$
Age 85	$\$5.234$	$\$4.832$	$\$4.480$
Table #5: Source IFID Centre Calculations			
GoMa Mortality $m = 86.34$, $b = 9.5$			

The same qualitative results follows when we increase the interest rate r from 4% to 6%, while maintaining $\lambda = 0$, $m = 86.34$ and $b = 9.5$. In this case $a_{65} = 10.474$, $a_{75} = 7.696$ and $a_{85} = 4.832$ respectively. This is identical to the impact of higher interest rates on the value of a (mortality free) fixed-income bond.

Finally, that if instead of using a GoMa value of $m = 86.34$ we increase the modal value to $m = 90$ and retain the dispersion parameter $b = 9.5$, the annuity factors increase to $a_{65} = 13.753$, $a_{75} = 10.094$ and $a_{85} = 6.434$ respectively. The higher values are obviously due to the longer lifespan. Under these parameters, the value of a pension annuity that pays $650 per month is $(13.753)(7800) = \$107,273$ at age 65, which is higher than the $100,000 premium displayed in Table #1.

With some numerical examples out-of-the-way, let us push the algebra one step further. By substituting $\lambda = 0$, the annuity factor can be simplified to:

$$\overline{a}_x = be^{((x-m)r+b\lambda(x))}\Gamma\left(-rb, b\lambda(x)\right). \tag{14}$$

This is the pure Gompertz (no Makeham) case. In fact, if we let $r = 0$ as well – which is the annuity factor when interest rates are zero – the equation collapses to an even simpler:

$$E[\mathbf{T}_x] = be^{b\lambda(x)}\Gamma\left(0, b\lambda(x)\right), \tag{15}$$

which oddly enough is the expected remaining lifetime under the GoMa law of mortality. Why is this so? Well, if you look at equation (3) you will see the seeds of this identity. Indeed, go ahead an "plug in" a value of $r = 0$ in equation (3) and you will see the definition of the ERL, which is: $E[T_x]$.

For example, under the same $m = 86.34$ and $b = 9.5$, when we compute the annuity factor under a zero interest rate we get $\overline{a}_{45} = 36.445$ years at age 45, $a_{55} = 27.189$ at age 55 and $\overline{a}_{65}$

$= 18.714$ at age 65. Recall, yet again, that implicit in the annuity factor (symbol) $\overline{a}_x$ is an interest rate r, as well as the GoMa parameters λ, m, b.

7 Deferred Annuities: Variation on a Theme

Imagine a situation in which you purchase a pension annuity at age y, but the contract stipulates that it does not start providing income until age $x > y$. Furthermore, if you don't actually survive to age $x > y$, you get nothing. Clearly, the value of this deferred annuity factor should be much less than $\overline{a}_y$ since you are not getting any income during the next $x - y$ years. Furthermore, the value should also be less than $\overline{a}_x$ since (i) there is a chance you will not survive to age x, and (ii) the insurance company has access to your premium during this time. In fact, when you combine these two elements, you are left with a **Deferred Pension Annuity Factor (DPAF):**

$$\overline{a}_y^{\tau} := \overline{a}_x({}_{\tau}p_y)e^{-r\tau}, \tag{16}$$

where $\tau = x - y$. In general, I will omit the τ superscript whenever $\tau = 0$ and the annuity factor is an immediate one so that $\overline{a}_x^0 := \overline{a}_x$.

Some might benefit from an alternative view in which DPAF is defined via:

$$\overline{a}_y^\tau = \int_\tau^\infty e^{-\left(rt+\int_0^t \lambda(y+s)ds\right)} dt, \tag{17}$$

which differs from the IPAF definition in equation (3) by virtue of the τ in the lower bound of integration, instead of zero and the IFM curve starting from $\lambda(y)$ instead of $\lambda(x)$. Indeed, the payments start at time τ, or age $y+\tau$, and therefore the "summation" of benefits must start at τ as well.

Under the GoMa law of mortality, the equation for the DPAF presented in equation (17) can again be "solved" in terms of the Incomplete Gamma function, leading to:

$$\overline{a}_y^\tau = \frac{b\Gamma\left(-(\lambda+r)b, \exp\{\frac{y-m+\tau}{b}\}\right)}{\exp\left\{(m-y)(\lambda+r) - \exp\{\frac{y-m}{b}\}\right\}} \tag{18}$$

:Here is a detailed numerical example. Assume the same GoMa parameters of $\lambda = 0$, $m = 86.34$, $b = 9.5$ and a valuation rate of $r = 4\%$. The expected remaining lifetime for a $y = 45$ year-old is $E[\mathbf{T}_{45}] = 36.46$ years, which also means that the expected age at death is 81.46 years. The probability that a $y = 45$ year-old survives twenty more years to age $x = 65$ is $({}_{20}p_{45}) = 0.911$ or a 91.1% chance. The time-value of money factor for 20 years under an $r = 4\%$ is $e^{-0.04(20)} = 0.449$, which is slightly less than fifty cents on the dollar. The immediate pension annuity factor at age $x = 65$ is $\overline{a}_{65} = 12.454$ per dollar of lifetime income. Finally, multiply these three numbers together and you get an age-45 "value" of $\overline{a}_{45}^{20} \approx (0.911)(0.449)(12.454) \approx \$5.1$ per dollar of lifetime income, starting at age 65.

A 45 year-old wants Lifetime Income of $1			
Deferred Pension Annuity Factor (DPAF) $\overline{a}_{45}^{\tau}$			
Starting at...	$r = 4\%$	$r = 6\%$	$r = 8\%$
Age $55, \tau = 10$	$\$10.354$	$\$6.804$	$\$4.597$
Age $65, \tau = 20$	$\$5.099$	$\$2.875$	$\$1.649$
Age $75, \tau = 30$	$\$1.964$	$\$0.951$	$\$0.465$
Age $85, \tau = 40$	$\$0.449$	$\$0.186$	$\$0.077$
Table #6: Source IFID Centre Calculations			
GoMa Mortality $m = 86.34$, $b = 9.5$			

8 Period Certain vs. Term Certain

Recall from the earlier chapter on modeling fixed-income bonds, that the value of a bond which paid a coupon yield of $c = 1$ dollar per annum until maturity...At the time we define a hybrid-bond....The **Term Certain Annuity Factor (TCAF):**

$$V(r,T) := V(1,r,T) - e^{-rT},$$

is the value of a coupon bond paying:

9 Duration of Pension Annuity

Akin to the concept of duration (and convexity) in the case of generic fixed-income bonds, we can define the same idea within the context of annuity factors. The duration D_x of the annuity factor is the (negative) derivative with respect to the valuation rate r, scaled by the annuity factor itself $\overline{a}_x$.The formal and explicit definition the annuity factor duration is:

$$D(x, \tau, r, \lambda, m, b) := -\frac{\partial}{\partial r}\overline{a}_x^{\tau}. \tag{19}$$

I use the same symbol D for duration, which was also used for generic fixed-income bonds, but with the understanding that the additional terms $(x, \tau, r, \lambda, m, b)$ will clarify the context in which the duration is intended. Note that mortality obeys a simple exponential distribution, the duration parameter can be easily computed as:

$$D(x, \tau, r, \lambda, m, b) = \frac{1}{r + \lambda}. \tag{20}$$

Oddly enough, in the case of exponential mortality the duration parameter D is equal to the annuity factor $\overline{a}_x$ itself! Thus, a small change in rates Δr will "move" the annuity factor by $-\Delta r \times \overline{a}_x$.

More generally, under a GoMa law of mortality the calculus doesn't work out as nicely and the expression for $D(x, \tau, r, \lambda, m, b)$ is a "mess". Luckily, we are able to obtain some (numerical) values by taking derivatives symbolically and evaluating the results. The following Table provides some examples.

The Duration Value $D = -\frac{\partial}{\partial r}\overline{a}_x$			
Immediate Pension Annuity Factor (IPAF)			
Starting at:	$r = 4\%$	$r = 6\%$	$r = 8\%$
Age 55	11.76 yrs	10.26 yrs	8.99 yrs
Age 65	9.13 yrs	8.21 yrs	7.39 yrs
Age 75	6.49 yrs	5.99 yrs	5.55 yrs
Age 85	4.1 yrs	3.88 yrs	3.68 yrs
Table #7: Source IFID Centre Calculations			
GoMa Mortality $m = 86.34$, $b = 9.5$			

What about convexity? You can go thru an even "messier" exercise to compute the second derivative of the annuity factor:

$$K(x, \tau, r, \lambda, m, b) := \frac{\partial^2}{\partial r^2}\overline{a}_x^{\tau},$$

which can in fact be computed symbolically using a variety of computer languages. Some examples of convexity values are $K = 195.\,497$, when $x = 55, r = 5\%$ and $\lambda = 0, m = 86.34$, $b = 9.5$ for the GoMa parameters. But, when $x = 45$ and $\tau = 10$ under the same valuation rate $r = 5\%$, the value of convexity is $K = 515.11$. The numbers are different but the pattern is the same as before. Longer deferral period increase both the duration and the convexity of the annuity factor.

Table #8 provides a summary picture of the duration and convexity values for various

Pension Annuity Factor at age $x = 50$ and $r = 5\%$			
Deferral Period	Value	Duration	Convexity
0 yrs	$\$15.229$	12.058 yrs	237.23
10 yrs	$\$7.477$	19.839 yrs	453.15
20 yrs	$\$3.087$	27.439 yrs	787.19
30 yrs	$\$0.895$	35.073 yrs	1246.84
Table #8: Source The IFID Centre Calculations			
GoMa Mortality $m = 86.34$, $b = 9.5$			

10 Variable vs. Fixed Pension Annuities

An investment of W premium dollars into an immediate *variable* annuity will entitle the annuitants to a lifelong payment of $W/\overline{a}_x$ *units* per year as opposed to dollars per year. As before,

$$\overline{a}_x := \int_0^\infty e^{-ht}({}_tp_x)dt, \tag{21}$$

but in this case the valuation rate h is rather arbitrary. You will see why in a moment, but for now note that it is often called the *assumed interest rate* (AIR) in the insurance lexicon.

Each payment unit entitles the individual to a variable (i.e., random) payment that depends on the performance of the chosen underlying asset (typically, an equity fund) *vis a vis* the AIR h. If the return on the underlying asset in any one period is less than the AIR h, the variable payment will decrease. If, on the other hand, the return on the asset is greater than the AIR h, the variable payment will increase. Formally, if the underlying asset is the same risky asset whose price dynamics are governed by a Brownian motion, then the immediate variable annuity's dollar income at time t, will be:

$$\frac{W}{a_x}\exp\{(\nu - h)t + \sigma B_t\}, \tag{22}$$

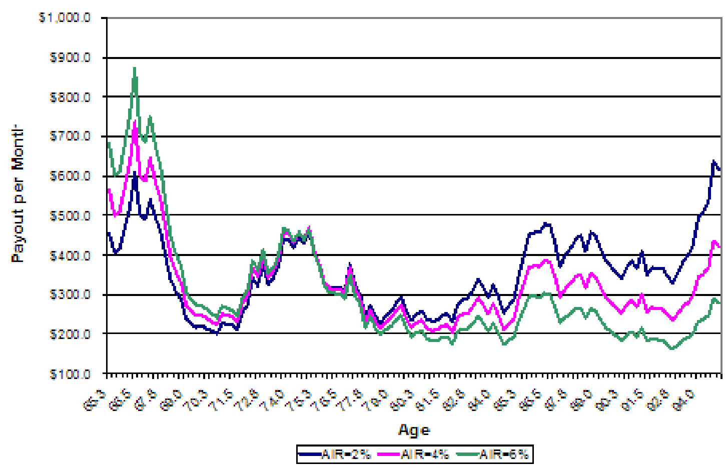
where B_t, ν, σ were all defined earlier.

For example, in the case of exponential mortality, $a_x =$

$1/(\lambda + h)$ and the income flow becomes:

$$(\lambda + h)W \exp\{(\nu - h)t + \sigma B_t\}. \tag{23}$$

The expression for this variable annuity income may seem obscure at first, but a comparison to the income from a fixed immediate annuity is quite illustrative. For example, if the AIR h is equal to the valuation rate (i.e., $h = r$), the individual is entitled to $(\lambda + r)W$ units. If the chosen underlying asset were a risk-free asset, then $v - h = 0$ and $\sigma = 0$, and so each unit would pay-off \$1 per year. Therefore, the total income would be exactly the same as in the fixed immediate annuity case: $(\lambda + r)W$ per year for life. The following figure illustrates how this would work.



2.One Sample Path - Three Outcomes Depending on h.