# What is a Sustainable Spending Rate?<br>A Simple Answer<br>(That Doesn't Require Simulation)

By: Moshe A. Milevsky, Ph.D.[1]
Finance Professor, York University
Executive Director, The IFID Centre
Toronto, Canada

## Abstract

*A number of financial commentators have emphasized the need for more research on sustainable spending rates from diversified portfolios – see, for example, Arnott (2004) -- a topic which is of relevance to individual retirees as well as most large foundations and endowments. Motivated by this apparent gap in the literature, I provide a simple analytic formula for the probability that a portfolio earning a lognormal investment return subject to a constant (after-inflation) withdrawal rate will be sustainable over a random lifetime horizon. The formula parsimoniously meshes asset allocation parameters, mortality estimates and spending rates without resorting to opaque and often irreproducible Monte Carlo simulations. I demonstrate how the biological aging process can be mapped into the mean and variance language of investment theory. In this framework, increasing the force of mortality is equivalent to reducing portfolio variance and increasing portfolio returns. Among the practical insights emanating from this approach, I confirm that a 65-year old retire with a stochastic lifetime horizon faces a 10% chance of ruin if he/she consumes more than $4-per-$100 principal of an equity-based portfolio with an expected real return of 7% and volatility of 20%. The insights obtained from this article will hopefully inspire financial analysts to provide investment guidance to the oncoming wave of retired baby boomers on more than just asset allocation matters.*

[1] Dr. Milevsky can be reached via Email at milevsky@yorku.ca or via Tel: (416) 348-9710 x 3010 or Fax: (416) 631-0916. Hardcopy address: The IFID Centre at the Fields Institute, 222 College Street, 2nd Floor, Toronto, Ontario, M5T 3J1, Canada. The author would like to thank Jin Wang and Anna Abaimova for research assistance.

> "…*Retirees Don't Have to Be So Frugal: Here is a Case for Withdrawing up to 6% a Year…" Jonathan Clements, Wall Street Journal, Page C1, November 17, 2004*
>
> *"…Forget the traditional approach and instead plan on withdrawing a fixed 5% of your portfolio's beginning-of-the-year value. Call it the take-five strategy…" Jonathan Clements, Wall Street Journal, Page C1, October 12, 2003*

## Section #1: MOTIVATION

Retirees – like endowment and foundation trustees -- share a similar dilemma. In addition to the classical asset allocation decision they must set an appropriate spending rate from their investment fund that will last forever (for endowments) - or at least during their random future lifetime (for retirees.)

Sustainable withdrawal and spending rates have been the focus of some academic research over the years. But, the topic has developed a renewed sense of urgency as a wave of North American baby boomers approaches retirement and seeks wealth management guidance on "what's next" for their IRA or 401(k) plan. Yet, for endowments and foundations, this topic has a 30-year history going back to a special session at the *American Economics Association* devoted to spending rates in which Tobin (1974) cautioned against consuming anything other than dividends and interest income[2], which in today's environment doesn't amount to much. Around the same time -- within the context of a private foundation or endowment -- Ennis and Williamson (1976) were the first to jointly analyze an appropriate asset allocation in conjunction with a given spending policy. More recently, Altschuler (2002) has argued that endowments are actually "too stingy" and not spending enough, while Dybvig (1999) has

[2] According to data compiled by the *National Association of College and University Business Officers* (NACUBO) endowment survey, the median spending rate in 2003 was 5% of assets, with the 10th percentile being 4.0% and the 90th percentile being 6.4% of assets.

discussed how asset allocation can be used to protect a desired level of spending using a pseudo *portfolio insurance* scheme.

In the parallel retirement planning arena, many authors such as Bengen (1994), Ho, Milevsky and Robinson (1994), Cooley, Hubbard and Walz (1998, 2003), as well as Pye (2000) and Ameriks, Veres and Warshawsky (2003) have run extensive computer simulations – motivated by the game of life simulations envisioned by Markowitz (1991) -- to locate "prudent" spending rates. These results usually advocate withdrawals in the range of 4% to 6% of <u>initial capital</u> depending on age and asset allocation.

The problem with these and similar Monte Carlo based studies is that they (i) are extremely difficult to replicate, (ii) are quite time consuming to generate if done properly using the required number of simulation, and (iii) provide very little financial or pedagogical intuition on the tradeoff between risk and return[3].

Along these lines, a recent article in the *Financial Analysts Journal* by Arnott (2004) claims that "…our industry pays scant attention to the concept of sustainable spending which is key to effective strategic planning for corporate pensions, public pensions, foundations and endowments – even for individuals…"

Therefore, partially driven by the "call" for more research in this area, in this paper I address the issue from a different – and what I believe is a novel -- perspective. My pedagogical objective is to shed light on the "financial intuition" of spending rates by deriving a simple analytic relationship between spending and sustainability in a random environment. Namely, I introduce the analytic concept of a *stochastic present value* (SPV) and use this to provide an expression for the probability that an initial corpus (nest egg) will be depleted under a fixed consumption rule, when both rates of return and horizons are stochastic. I stress the dual uncertainty for returns and horizons, which is something that has not received much attention in the portfolio management literature, as it pertains to retirees.

[3] I have run some case-study examples using the 6, or so, free web-based simulators and have found a wide variation between the suggested "nest eggs" needed to support a comfortable retirement. A similar theme – which was misinterpreted as a criticism of the Monte Carlo method -- was echoed in a recent Bloomberg Wealth Manager article by Ed McCarthy (Dec2002/Jan2003), page 39-54.

And, in contrast to almost all other papers and authors that have tackled this problem, I do not resort to any forward-looking Monte Carlo Simulations to locate “prudent” spending rates. Rather, I base the analysis on the above-mentioned SPV and a continuous-time approximation under lognormal returns and exponential lifetimes. In the case of an investor with an infinite horizon (perpetual consumption), this formula is exact. In the case of a random future lifetime, the formula is based on moment matching approximations which target the first and second moment of the “true” stochastic present value. The results are remarkably accurate when compared against more costly and time consuming simulations.

I also provide several numerical examples to demonstrate the versatility of the closed-form expression for the *stochastic present value* (SPV) in determining sustainable withdrawal rates and their respective probabilities. This formula can easily be implemented in Excel or any other spreadsheet using a variety of portfolio risk/return parameters, ages and withdrawal rates, and reproduces results that are within 5% of extensive Monte Carlo Simulations.

The remainder of this paper is organized as follows. Section 2 casts the mathematics of the *sustainable spending* problem within the context of a traditional present value of future cash-flows calculation. Section 3 provides a closed-form analytic expression for the probability that a given spending rate is sustainable. Section 4 provides extensive numerical examples over a variety of ages and spending rates. Section 5 concludes the paper and a technical appendix provides additional stress-testing results on an alternative investment return generating process.

## Section #2: STOCHASTIC PRESENT VALUE (SPV) of SPENDING

Imagine that you plan to invest your money in a portfolio earning {*R%*} per annum and you plan to consume a fixed real (after inflation) dollar each and every year until some horizon denoted by {*T*}. If the horizon and investment rate of return are known with absolute certainty, the present value (PV) of your consumption at initial time zero would be computed via:

$$PV = \sum_{i=1}^{T} \frac{1}{(1+R)^i} = \frac{1-(1+R)^{-T}}{R}\,, \qquad \text{(eq.1)}$$

which is the textbook formula for an *ordinary simple annuity*, which should be familiar to all students of business and finance. Thus – in a deterministic world -- if you start retirement with a nest egg greater than the *PV* in equation (eq.1) times your desired consumption, your money will last for the rest of your life. If you have less than this amount you will be "ruined" at some age prior to death. For example, at an {*R=7%*} annual investment return and a {*T=25*} year horizon, the required nest egg is 11.65 (the PV in (eq.1)) times your real consumption. If you have more than this stock of wealth at retirement your plans are sustainable. However, if you start your retirement years with only 10 times you desired real consumption, then you will "run out of money" in precisely 17.79 years – an income gap of 7.2 years -- because the present value of $1 annuity for 17.79 years at 7% interest is $10. Note that as {*T*} goes to infinity – which we call the endowment case – the *PV* converges to the number {*1/R*}. At {*R=0.07*} the resulting PV is 14.28 times the desired consumption.

Of course, human beings have a random (and finite) lifespan and any exercise that attempts to compute required *present values* at retirement must account for this uncertainty. Table #1 and the corresponding Figure #1 illustrate the probabilities of survival using mortality estimates from the U.S.-based Society of Actuaries. A 65 year-old female has a 34.8% chance of living to age 90. A 65 year-old male has a 23.7% chance of living to age 90. The probabilities of survival decline in a roughly exponential manner with age, reaching close to zero sometime between ages 105 and 115 depending on the mortality table, projection method and gender. And, while the often quoted statistic for life expectancy is somewhere between 78 and 82 years in the U.S., this is only relevant at the time of birth. By the time pensioners reach their retirement years, they may be facing 25 to 30 more years with substantial probability.

From our retirement spending perspective, a 65-year-old might live 20 more years or 30 more years or only 10 more years. How should this impact the withdrawal rate?

**Table #1 and Figure #1 Placed Here.**

Should a 65-year-old plan for the 75$^{th}$ percentile, 95$^{th}$ percentile of the end of the mortality table? What {*T*} value should be used in (eq.1)? The same question applies to the investment return {*R*}. What is a reasonable number to use? The average real (after inflation) investment returns from a broadly diversified portfolio of equity during the last 75 years has been in the vicinity of 6% to 9% according to Ibbotson Associates (2004), but the year-by-year numbers can vary widely.

The correct approach, arguably, is not to guess, assume or take point-estimates but to actually account for this uncertainty within the model itself. Nobel laureate Bill Sharpe has amusingly called the (misleading) averaging approach with fixed returns and fixed dates of death, "financial planning in fantasy-land."

So, in contrast to the trivial deterministic case -- where both the horizon and the investment return are known with certainty -- when both of these variables are stochastic, the analogue to (eq.1) is a *stochastic present value* (SPV) defined by:

$$SPV = \frac{1}{(1+\tilde{R}_1)} + \frac{1}{(1+\tilde{R}_1)(1+\tilde{R}_2)} + \ldots + \frac{1}{\prod_{j=1}^{\tilde{T}}(1+\tilde{R}_j)} = \sum_{i=1}^{\tilde{T}}\prod_{j=1}^{i}(1+\tilde{R}_j)^{-1} \quad \text{(eq.2)}$$

where the new variable $\{\tilde{T}\}$ denotes the random time of death (in years), and the new $\{\tilde{R}_j\}$ denotes the random investment return in year *j*. Without any loss of generality $\{\tilde{T}=\infty\}$ is the infinitely lived endowment or foundation situation. Likewise, if the consumption/withdrawals take place once per month or once per week, the random variables $\{\tilde{R}_j\}$ and $\{\tilde{T}\}$ are adjusted accordingly. And, if the return frequency is infinitesimal, the summation sign in (eq.2) converges to an integral, while the product sign is converted into a continuous-time diffusion process.

The intuition behind (eq.2) is as follows. Looking forward, we must sum-up a random number of terms in which each denominator is also random. The first item discounts the first year of consumption at the first year's random investment return. The second item discounts the second year's consumption (if the individual is still alive) at the product of the first and second years' random investment return, etc.

The SPV defined by (eq.2) can be visualized in Figure #2. One can think of the stochastic present value as a random variable with a probability density function (PDF) that depends on the risk/return parameters of the underlying investment generating process as well as the random future lifetime. If we start with an initial endowment or nest egg of \$20 and intend to consume \$1 (after-inflation) per annum, the probability of sustainability is equal to the probability that the SPV is less than \$20. This corresponds to the area under the curve to the left of the ray emanating from \$20 on the x-axis. The probability of ruin is the area under the curve to the right of the \$20 ray. The precise shape and parameters governing the SPV depend on the investment and mortality dynamics, but the general picture is remarkably consistent and similar to Figure #2. This "family" of SPVs is defined over positive numbers, is right-skewed and is equal to zero, at zero.

**Figure #2 Placed Here.**

The four distinct curves in Figure #2 denote differing random life-spans. In the first, the (unisex) individual is 50 years old, in the second he/she is 60, in the third - 65 and in the last one - 75. As the individual ages, the SPV of future (planned) consumption shifts toward the left – relative to the same \$20 mark – since the odds are that \$20 is enough to sustain this standard of living when starting at an older age.

Now, we move on to our goal of obtaining a closed-form expression for the distribution of the SPV. It is quite common in financial economics (and especially option pricing theory) to assume that investment returns are generated by a LogNormal distribution, a.k.a. the geometric Brownian motion diffusion process. On a theoretical level this assumption has many supporters - from Merton (1975) to Rubinstein (1991). Empirically, however, I admit that it does not fit high-frequency data or observed returns over all time horizons. Interestingly, though, in a recent paper by Levy and Duchin (2004) the Log-Normal assumption actually "won" many of the "horse races" when comparing plausible distributions for historical returns. Furthermore, many popular optimizers, asset allocation models and often-quoted common advice are based on the classical Markowitz/Sharpe assumptions of Log-Normal returns. Therefore, for the remainder of this paper I will follow this tradition and shift the discussion on the impact of alternative assumptions to the appendix.

## Section #3: ANALYTIC FORMULA: SUSTAINABLE SPENDING

### BACKGROUND:

Before I come to the main part of the story, I must review three important probability distributions that play a critical role in the sustainability calculations. The first is the ubiquitous Log-Normal (LN) distribution, the second is the Exponential Lifetime (EL) distribution and the third and final one is the – perhaps lesser know -- Reciprocal Gamma (RG) distribution. The connection between these three will become evident in time.

**Log-Normal Random Variable:** The investment *total return* denoted by $\{R_t\}$ between time zero and time *t*, is said to be Log-Normally distributed with parameters $\{\mu,\sigma\}$ if the expected total return is $\{E[R_t]=e^{\mu t}\}$, the logarithmic volatility is $\{E[SD[\ln[R_t]]]=\sigma\sqrt{t}\}$ and the probability law can be written as $\{\Pr[\ln[R_t]<x]=N((\mu-0.5\sigma)t,\sigma\sqrt{t},x)\}$, where $\{N(.)\}$ denotes the cumulative normal distribution. For example, a mutual fund or portfolio that is expected to earn an inflation-adjusted continuously compounded return of $\{\mu=7\%\}$ per annum, with a logarithmic volatility of $\{\sigma=20\%\}$ has a $\{N(0.05,0.20,0)=40.13\%\}$ chance of earning a negative return in any given year. But, if the expected return is a more optimistic 10% per annum, the chances of losing money are reduced to $\{N(0.08,0.20,0)=34.46\%\}$. Note that while the expected value of the Log-Normal random variable $\{R_t\}$ is $\{e^{\mu t}\}$, the median value (a.k.a. geometric mean) is a lower $\{e^{(\mu-0.5\sigma^2)t}\}$. By definition, the probability a Log-Normal random variable is less than its median value is precisely 50%. The gap between the

expected value $\{e^{\mu t}\}$ and the median value $\{e^{(\mu-0.5\sigma^2)t}\}$ is always greater than zero, proportional to the volatility and increasing in time.

**Exponential Lifetime Random Variable**: The future lifetime random variable denoted by the letter *T* is said to be exponentially distributed with mortality rate $\{\lambda\}$ if the probability law for *T* can be written as: $\{\Pr[T > s] = e^{-\lambda s}\}$. The expected value of the Exponential Lifetime random variable is equal to and denoted by $\{E[T] = 1/\lambda\}$ while the median value – which is the 50% mark -- can be computed via: $\{Med[T] = \ln[2]/\lambda\}$. Note that the expected value is greater than the median value. For example, when $\{\lambda = 0.05\}$ the probability of *living* for at least 25 more years is: $\{e^{-(0.05)(25)}$ = 28.65%}, and the probability of living for 40 more years is: $\{e^{-(0.05)(40)}$ = 13.53%}. The expected lifetime is $\{1/0.05$ =20} years and the median lifetime $\{\ln[2]/0.05$ =13.86} years. The exponential assumption is a very convenient one for future lifetime random variables. And, although human aging does not quite conform to an exponential – or constant force of mortality – assumption, I will show that for the purposes of estimating a sustainable spending rate, it does a remarkably good job of capturing the salient features.

**Reciprocal Gamma Random Variable:** A random variable denoted by *X* is said to be Reciprocal Gamma distributed with parameters $\{\alpha, \beta\}$ if the probability law for *X* can be written as:

$$\Pr[X < x] := \frac{\beta^{-\alpha}}{\Gamma(\alpha)} \int_0^x y^{-(\alpha+1)} e^{(-1/y\beta)} dy \qquad \text{(eq.3)}$$

One need not be intimidated by the somewhat messy-looking integral since knowledge of calculus is not required to actually use the formula. The cumulative distribution function (CDF) displayed in equation (eq.3) plays the same role as the CDF of the Normal or Log-Normal distribution, both of which are now ubiquitous in finance. The definition of the Reciprocal Gamma random variable is such that the probability an RG random variable is *greater* than x is equivalent to the probability that a Gamma random variable is *less* than $\{1/x\}$. The CDF of a Gamma random variable is available in all statistical packages -- even in Excel – and thus should be easily accessible to most readers.

Finally, the expected (mean) value – a.k.a. first moment -- of the Reciprocal Gamma distribution is $\{E[X]=(\beta(\alpha-1))^{-1}\}$ and the second moment is $\{E[X^2]=(\beta^2(\alpha-1)(\alpha-2))^{-1}\}$. For example, within the context of this paper a typical parameters pair is: $\{\alpha=5,\beta=0.03\}$. In this case, the expected value of the Reciprocal Gamma variable is $\{1/((0.03)(4))$=10}. The probability the Reciprocal Gamma random variable is greater than 8, for example, is 45.62%. In contrast, if we increase $\{\alpha\}$ from a value of 3 to a value of 4, the relevant expected value becomes $\{E[X]$=6.66} and the probability is $\{\Pr[X>8]$=24.24%}

## THE MAIN RESULT: EXPONENTIAL RECIPROCAL GAMMA (ERG)

With the mathematical background behind us, my primary claim is that if one is willing to assume Log-Normal returns in a continuous time setting then the stochastic present value – the one displayed graphically in Figure #2 – is actually Reciprocal Gamma distributed in the limit. In other words, the probability that the SPV is greater than the initial wealth or nest egg denoted by $\{w\}$, is the simple-looking:

$$\Pr[SPV>w]=\text{GammaDist}\left(\frac{2\mu+4\lambda}{\sigma^2+\lambda}-1,\frac{\sigma^2+\lambda}{2}\,|\,\frac{1}{w}\right) \qquad \text{(eq.4)}$$

where $\text{GammaDist}(\alpha,\beta\,|\,.)$ denotes the cumulative distribution function (CDF) of the Gamma distribution – using the Microsoft Excel notation – evaluated at the parameter pair $\{\alpha,\beta\}$. The familiar $\{\mu,\sigma\}$ are the expected return and volatility parameters from the investment portfolio and $\{\lambda\}$ is the mortality rate. The expected value of the SPV – based on the Reciprocal Gamma representation – is $\{(\mu-\sigma^2+\lambda)^{-1}\}$.

For example, start with an investment (endowment, nest egg) fund containing $20 that is invested in an equity fund that is expected to earn $\{\mu=0.07\}$ per annum, with a volatility or standard deviation of $\{\sigma=0.20\}$ per annum. Assume that a (unisex) 50-year-old with a median future lifespan of 28.1 years -- according to Society of Actuaries mortality tables -- intends on consuming $1 after-inflation per annum for the rest of his or her life.

Recall that if the median lifespan is 28.1 years, then by definition the probability of survival for 28.1 years is exactly 50%, which implies that our "mortality rate" parameter is: $\{\lambda = \ln[2]/28.1 = 0.0247\}$. According to (eq.4) the so-called probability of retirement ruin, which is the probability that the stochastic present value of $1 consumption is greater than $20, is approximately 26.8%. In the language of Figure #2, if we evaluate the SPV at {*w=20*}, the area to the right has a mass of 0.268 units. The area to the left – which is the probability of sustainability – has a mass of 0.732 units. Naturally, different values of {w} will result in different ruin probabilities.

The more technically inclined readers might want more than a formula. Indeed, a proof that (eq.4) is the distribution of the stochastic present value is based on moment matching techniques and the Partial Differential Equation (PDEs) for the probability of ruin. A variant of this result can actually be traced back to a paper by Merton (1974), although it was never exploited in the context of spending rates. For more details, proofs and restrictions, see Milevsky (1997) or Browne (1999) and the references contained therein.

In addition, the appendix contains a brief description of tests that where conducted to stress-test the formula in equation (eq.4) under a variety of alternative asset-return dynamics, and specifically the implications of assuming an extreme pure jump process. Overall, the formula survives the various litmus tests provided the parameters are within the region of normal retirement. The following section displays extensive numerical results.

## Section #4: DETAILED NUMERICAL EXAMPLES:

A newly retired 65 year-old has a nest egg of $1,000,000 which must provide income and last for the remainder of this individual's natural life. In addition to expected Social Security benefits of $14,000 per annum and a defined benefit (DB) pension from an old employer providing $16,000 per annum – both payments adjusted for inflation each year – the retiree estimates the need for an additional $60,000 from the investment portfolio. The $60,000 income will be coaxed from the million dollar portfolio via a systematic withdrawal plan (a.k.a. SWiP) that sells-off the required number of shares/units each month using a reverse dollar-cost average (DCA) strategy. All of these numbers are prior to any income taxes. Nor do I distinguish between tax sheltered (IRA, 401k) plans versus taxable plans, which are a different set of important issues I do not address in this paper. What is important to note is that the $90,000 consumption plan will be satisfied with $30,000 from a *de facto* inflation–adjusted life annuity and the remaining $60,000 from a SWiP.

In our previous lingo, I am interested in whether the stochastic present value (SPV) of the desired $60,000 income per annum is *probabilistically less* than the initial nest egg of one million dollars. If this is the case, the standard of living is sustainable. If, however, the SPV of the consumption plan is larger than one million dollars, the retirement plan is deemed sustainable and the individual will be *ruined* at some point in their life, unless they reduce their consumption habits. The basic philosophy of this paper is that the SPV is a random variable and the proper analysis comes down to probabilities.

**Table #2a, #2b, #2c Placed Here**

Table #2 provides an extensive combination of consumption/withdrawal rates across various ages so readers can gauge the impact of these factors on the ruin probability. The first column displays the retirement age the second column displays the median age-at-death based on actuarial mortality tables and the third column computes the implied hazard rate from this median value. With a $\{\lambda\}$ value in hand and the $\{\mu\}$ and $\{\sigma\}$ given in the lower-left corner, the table evaluates the SPV of various spending rates ranging from $2 to $10.

The first row within table #2a provides results in the case of a retiree who would like the spending to last forever – and hence the median age at death is infinity – which is also applicable to an endowment or foundation with an infinity horizon. The probability of ruin ranges from a low of 15% ($2 spending) to a high of 92% ($10 spending) when the portfolio is invested in an equity-based portfolio that is expected to earn a (lognormal) investment return with a mean value of $\{\mu=7\%\}$ and a volatility of $\{\sigma=20\%\}$ per annum.

Back to our retiree, according to Table #2a, if the 65 year-old invests the one million dollar nest egg in the same equity-based portfolio, the exact probability of ruin – i.e. the probability the plan is not sustainable – is 25.3%. Roughly one out of four retirees who adopt this retirement consumption plan will be forced to reduce their standard of living during retirement. By exact probability of ruin, I mean the outcome from discounting all future cash-flows using the correct (unisex) actuarial mortality table starting at age 65.

In the same table, just above the exact 25.3% number, I list the results using the ERG approximation formula, which is based on an exponential future lifetime implemented within equation (eq.4). Note the approximate answer is a slightly higher 26.2% probability of ruin, relative to the 25.3% under the exact method. The gap between the exact and approximate number is less than 0.9% which provides additional confidence in our ERG formula (eq.4)

Now, I would argue that regardless of whether one uses the exact or the approximate methodology, a 25% chance of retirement ruin, which is only a 75% chance of success, should be unacceptable to most retirees. Table #2a indicates that lowering the desired consumption or spending plan by $10,000 to a $50,000 SWiP reduces the probability of ruin to 16.8% (using the exact method) or 18.9% (using the approximation). And, if the spending plan is further reduced to $40,000 the probability of ruin shrinks to 9.4% (exact) and 12.3% (approximate). The retiree – and his or her financial planner or analyst – can determine whether these odds are acceptable *vis a vis* their tolerance for risk.

In the other direction, if the same individual were to withdraw (the entire) $90,000 from the million dollar portfolio – using the 7% mean and 20% volatility portfolio parameters – the probability of ruin would be 50.5% (exact) or 48.3% (approximate).

To understand the intuition behind the numbers, note that the mean or expected value of the stochastic present value (SPV) of $1 of real spending is: $\{1/(\mu-\sigma^2+\lambda)\}$, where $\{\mu\}$ and $\{\sigma\}$ are the investment parameters, while $\{\lambda\}$ is the mortality rate parameter induced by a given median future lifetime. For a 65 (unisex) year-old the median future lifetime is 18.9 years according to the RP2000 Society of Actuaries mortality table. To get the 50% probability point with an exponential distribution, we must solve the equation $\{e^{-18.9\lambda}=0.5\}$, which leads to $\{\lambda=\ln[2]/18.9=0.0367\}$ as the implied rate of mortality.

Now back to the mean value of the SPV, in the case of {$\mu$=7%} and volatility of {$\sigma$=20%}, this works out to {$1/(0.07-0.04+0.0367)$}, which is an average of $15 for the SPV per dollar of desired consumption. Thus, if the retiree intends on spending $90,000 per annum, it should come as no surprise that a nest-egg of only 11 times this amount is barely enough to give even odds. Note that the expected value of the SPV decreases in {$\mu, \lambda$} and increases in {$\sigma$}. While the impact of portfolio parameters should be obvious – higher mean is good, higher volatility is bad -- the benefit of a higher mortality rate {$\lambda$} comes from reducing the anticipated lifespan and hence the length of time over which the withdrawals are taken.

Now, if the same individual were to delay retiring by five years, or more precisely, begin consuming from the nest egg at age 70, the same $60,000 consumption plan would result in a 17.6% probability (exact) or 20.1% probability (approximate) of ruin according to the same Table #2a. The increased sustainability of the same plan – relative to the roughly 25% probability if this individual were to retire at age 65 – is due to the reduced future lifespan and hence the lower stochastic present value of consumption. Think back to the expected value of the consumption plan. At age 70 the median future lifespan is only 14.6 years, which leads to a higher {$\lambda = 0.0475$} and hence a lower value for E[*SPV*]. The retiree can start retirement with less or can consume more.

Table #2b and #2c provide results on alternative portfolio investment parameters using the ERG approximation from equation (eq.4). In Table #2b I have reduced the expected investment return from 7% to 5% but left the volatility at 20%. In this case all the probabilities are higher compared to Table #2a since a higher volatility can only make things worse. In Table #2c I have reduce the volatility from 20% to 10% and kept the expected return at 5%. For example, the 65 year-old withdrawing $60,000 from a million dollar portfolio has 39.8% probability of ruin under a {$\mu$=5%} and {$\sigma$=20%} investment regime, compared to a 26.2% probability of ruin under a {$\mu$=7%} and {$\sigma$=20%} investment regime, which is obviously due to the 200 basis point loss in returns. But, when the {$\mu$=5%} investment return is matched with (a more reasonable) {$\sigma$=10%} volatility, the probability of ruin shrinks to 21% according to Table #2c. The intuition once again comes down to the expected value of the SPV of $1 spending: {$1/(\mu-\sigma^2+\lambda)$}. When {$\mu$=5%} and {$\sigma$=10%} the first part of the denominator is

0.04, but when {$\mu$=7%} and {$\sigma$=20%} the same term is only 0.03, which *ceteris paribus* increases the SPV which lowers the sustainable spending rate. Note that Table #2c does not provide uniformly lower probabilities of ruin. For high levels of consumption a more aggressive {$\mu = 0.07, \sigma = 0.20$} portfolio may lead to better sustainability odds compared to the more conservative {$\mu = 0.05, \sigma = 0.10$} portfolio.

One can think of a number of ways in which to "play" with this formula. For example, our main (eq.4) can be inverted to solve for a "safe" rate for a given probability of ruin tolerance. This idea is akin to some recent applications of shortfall as a measure of risk in the context of portfolio management. See Browne (1997, 1999) or Young (2004) for more examples of this concept in a dynamic control framework. Thus, if one desires a 90% probability of sustainability when the median future lifetime is 15 years under a portfolio that is projected to earn {$\mu = 0.05$} with a standard deviation of {$\sigma = 0.10$}, then the consumption rate that leads to a 10% probability of ruin is $*5.03* per annum.

Along the same lines, the impact of the *expected return* {$\mu$} on the sustainability of a given withdrawal strategy can easily be "stress tested". For example, at 85% desired probability of sustainability the implied withdrawal rate is $4.41 per annum when the expected return is 7% (with a 20% standard deviation). However, if we remove 100 basis points from the equity return so that {$\mu = 0.06$}, the same 85% forces a more conservative $3.76 withdrawal rate.

Another interesting insight comes from examining the interplay between the parameters in our formula. If we reduce the fixed mortality rate {$\lambda$} by 100 basis points – which increases the median future lifetime from {$\ln[2]/\lambda$} to $\ln[2]/(\lambda + 0.01)$ -- it has the "probability equivalent" effect of increasing the portfolio return by 200 basis points and increasing the portfolio variance by 100 basis points. They both lead to the same statistical results. Recall that our {$\alpha, \beta$} parameter arguments in (eq.4) can be expressed as a function of {$\mu + 2\lambda$} and {$\sigma^2 + \lambda$}. Thus, having a longer lifespan (i.e. lower hazard rate) is interchangeable with decreasing the portfolio return and portfolio variance relative to the baseline. In aggregate, however, this increases the probability of ruin and reduces the probability that a given level of wealth is enough to sustain retirement spending.

Finally, it is important to stress that in the $\{\lambda = 0\}$ -- infinite horizon -- case our result is not an approximation. It is a theorem that the SPV is in fact Reciprocal Gamma distributed. For those readers who remain unconvinced that what is effectively the "sum of lognormals" in (eq.4) can converge to the inverse of a Gamma distribution, I urge you to simulate the SPV for a reasonably long horizon and conduct a KS goodness of fit test of the inverse of these numbers against the Gamma distribution, with the parameters given by $\{\alpha = (2\mu + 4\lambda)/(\sigma^2 + \lambda) - 1, \beta = (\sigma^2 + \lambda)/2\}$. As long as the volatility parameter $\{\sigma\}$ is not "too high" relative to the expected return $\{\mu\}$, we get convergence of the relevant integrand. Thus, it is only in the random lifespan where $\{\lambda > 0\}$ that our result is approximate, albeit correct to within two moments of the true SPV density. To illustrate this graphically, Figure #3 provides a stylized illustration -- under a 7% mean and 20% volatility -- of the approximation error from using the ERG formula based on an exponential future lifetime when in fact the "true" future lifetime random variable is more complicated.

**Figure #3 Placed Here**

Figure #3 displays the retirement ruin probability – a.k.a. the probability the spending rate is not sustainable – starting at age 65 for a range of consumption rates from $1 to $10 per original $100 nest egg. For low consumption rates the ERG formula slightly overestimates the probability of ruin and thus gives a more pessimistic picture of the sustainability of spending. At higher consumption rates the exact retirement ruin probability is higher than what is claimed by the approximation. Yet, notice the relatively small error gap between the two curves, which at their worst is no more than 3% - 5%. The two curves are at their closest – which implies that the approximation is at its best – when the spending rates is between $5 to $7 per original $100, which coincidently is precisely where the current debate regarding sustainable spending currently resides.

## Section #5: CONCLUSION AND NEXT STEPS

A casual search on the Web reveals close to a dozen on-line calculators -- most sponsored by financial services companies -- that purport to compute a sustainable withdrawal rate (and asset allocation) for retirees by using Monte Carlo Simulations. A number of these calculators are plagued by opacity in the details of their stochastic generating methodology, and conduct an absurdly small number of simulations when compared with the tens of thousands needed for convergence. Moreover, the uncertainty generated by the randomness of human life is often ignored or alluded to outside of the formal model. Indeed, the black-box and time consuming nature of obtaining results do little to enhance a pedagogical understanding of the withdrawal or spending problem. The same issues are relevant in the endowment business where trustees and other decision-makers must tradeoff current spending against future growth.

The distinction between traditional Monte Carlo simulations and the analytic techniques promoted in this paper is more than just a question of academic tastes and techniques[4]. While Monte Carlo simulations will continue to have a legitimate and important role within the field of wealth management and retirement planning, I believe that a simple, easy to use and baseline formula can serve as a sanity check or a calibration point for more complicated simulations. At the risk of overselling, this is akin to having a Black-Scholes formula for the price of a call or put option – when many of the underlying assumptions are questionable -- which provides a deep understanding of the embedded risk and return tradeoffs and can live side by side with more sophisticated simulation based option-pricing models.

For example, using the formula I find that a (unisex) 65 year-old retiree who invests his/her portfolio in a market that is expected to earn a real (after-inflation) 7% with a volatility of 20% and consumes $4 per-year per $100 of initial portfolio value, will get "ruined" 10 times out of 100. However, if the same retiree withdraws a more aggressive $6 per $100, the probability increases to about 25% or one time out of four. This is clearly not sustainable. As an upper bound, a retiree should be spending no more than $\{\mu - \sigma^2 + \lambda\}$ percent of the initial nest egg, where $\{\mu\}$ is the expected return, $\{\sigma\}$ is the volatility and $\{\lambda = \ln[2]/m\}$, where *m* is a median future lifetime. This spending rate would be sustainable "on average" but not much better.

[4] See, for example, the recent *Wall Street Journal* article (August 31, 2004) entitled: "Tool Tells How Long Nest Egg Will Last", in which the reporter Kaja Whitehouse described the benefits of analytic PDE-based solutions over Monte Carlo simulations.

Note that most of these numbers are in-line with results from a variety of simulation studies – for example the widely used Ibbotson Associates retirement wealth simulator -- albeit produced by an insightful formula in a fraction of the time. Future and ongoing research is examining the impact of income taxes and optimal location decisions as well as well as the role of life annuities in increasing the sustainability of a given spending rate.

Our hero, of course, is the (Reciprocal) Gamma distribution, which should take its rightful place beside the Log-Normal density in the pantheon of probability distributions that are of immediate relevance to financial practitioners and portfolio managers.

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## Appendix 1: Alternative Process for Investment Returns

The simulations and analytic results within the main body of the paper are all predicated on Log-Normally distributed portfolio returns – a.k.a. GBM diffusion -- under which (i) the mean and standard deviation of investment returns completely specify the return dynamics and (ii) the portfolio wealth process is continuous in time. In this appendix I conduct the same *ruin probability* calculations under various ages and spending rates, but assuming the underlying investment returns follow a pure discontinuous jump process. My objective is to stress test the robustness of the main formula in equation (eq.4).

Indeed, the discontinuity over time – a.k.a. jumps -- for both individual stocks and aggregate index values has been extensively researched in the empirical finance literature for over ten years. In addition, the Log-Normal assumption has fallen into somewhat of disrepute amongst option pricing specialists who have documented time-varying (i.e. non constant) volatilities for most traded financial instruments. For example, Bates (2003) develops and calibrates a diffusion-based model for option pricing under which the return generating process is a mixture of Brownian movements and Poisson jumps. A large number of recent papers in the mathematical finance literature have adopted a mixture approach. And, it is most common in the derivative securities hedging arena. One might therefore rightfully expect to see a more sophisticated model for security prices in a 21st century paper. However, in this paper I am not interested in locating the most precise model for the fine structure of investment returns. Rather, I am interested in a *reasonable approximation* for long-term investment returns to help shed light on a reasonable spending rate. This core issue I would like to address is whether jumps and discontinuities in market prices have a meaningful impact on sustainable spending rates implied from a model that assumes a Log-Normal process. I am not interested in hedging derivative securities or computing Value-at-Risk (VaR) over two week horizons. Rather, I am concerned with 20 to 30 year forecasts where parameter uncertainty – i.e. what is the equity premium going forward – might swamp the model uncertainty.

I approach this question in the form of a classical "horse race" where I simulate two distinct time series, one being a pure diffusion and one being a pure jump process, and compute the ruin probabilities for various spending rates.

To make this analysis meaningful – or an apples to apples comparison -- I impose some structure on the jump process. I force the first two moments of the simulated pure jump process to equal the first two moments of a pure diffusive process. This moment matching procedure is often used in finance when the true return generating distribution is unknown but the moments are known with reasonable certainty. For example, we might expect markets to earn a 7% per annum with standard deviation of 20%, but we are not sure about the exact return generating process. More specifically and for the purposes of this appendix we calibrated parameters for the simulated jump process so that the (real, after inflation) arithmetic mean return was 7% (geometric mean return was 5%) and the volatility of returns was 20%.

To formally set-up the horse race I start with a generic investment return process $\{S_t\}$, defined in an exponential manner by the equation:

$$S_t = S_0 e^{X_t} . \tag{a.1}$$

The process $\{X_t\}$ in the exponent can either be the familiar and continuous:

$$X_t^{\ B} := gt + \sigma B_t , \tag{a.2}$$

or the purely discontinuous process:

$$X_t^{\ Y} := gt + \sum_{i=1}^{N_t(c)} Y_i , \tag{a.3}$$

One can recognize the former process for $\{X_t\}$ – equation (a.2) -- as the standard Brownian motion plus a linear trend *gt*. The latter process – equation (a.2) – is a random sum $\{N_t(c)\}$ of discrete jumps of random size $\{Y_t\}$. The $\{N_t(c)\}$ denotes a generic Poisson arrival process at a rate of *c* jumps per annum. The {} can be described heuristically as a process that counts the total number of jumps that have occurred prior to time t. It starts at zero and then increases by one unit eac random arrival time. By definition of the Poisson process, the time between jumps is exponentially distributed with a mean time of $\{1/c\}$ years. Finally, the probability density function for each jump $\{Y_t\}$ is assumed to be:

$$f(y) := \frac{1}{2} k e^{ky} I_{\{y<0\}} + \frac{1}{2} k e^{-ky} I_{\{y>0\}} , \tag{a.4}$$

which is symmetric around zero but exponential-like in character on both sides of zero. Thus, the exponential distribution plays two distinct roles in our pure-jump simulation process. First, it determines when jumps occur and second it determines the magnitude of those same jumps.

Of course, equations (a.3) and (a.4) are one of many possible ways to generate and model jumps for the investment return process – all of which are part of the Levy family for $\{X_t\}$ – and I have selected this particular parameterization for analytic convenience and its ability to capture the essence of jump processes in continuous time.

Actual simulation sample paths for the process $\{X^Y{}_t\}$ are generated in three stages. First an exponential random variable with parameter *c* – for example an average of 3 jumps per year -- generates a jump time. Then, a fair coin toss determines whether any given jump $\{Y_t\}$ is positive or negative. Finally an exponential random variable with parameter k -- for example $\{k = 8\%\}$ per jump -- generates a magnitude. The exponential distributions can be simulated using the Inverse Transform algorithm $\{-\ln[U]/k\}$ or $\{-\ln[U]/c\}$ applied to a uniformly distributed random variable *U*. See Ross (1997) for a detailed reference on standard probability simulations. The three-step procedure is repeated as simulated calendar time increases until the end of the desired sample path.

Recall that our objective is to compute the probability the portfolio value process defined by the stochastic differential equation (SDE):

$$dW_t = dS_t - dt \,, \qquad \text{(a.5)}$$

will hit a zero level (a.k.a. ruin) prior to death using both candidates for the exponential process $\{X_t\}$, but that they share the same first and second moments. Note that the complete symmetry of the jump-process density in equation (a.4) leads to a first-moment:

$$E[X^B{}_t] = E[X_t^{\,Y}] = gt \,, \qquad \text{(a.6)}$$

where the parameter *g* is the familiar geometric mean return. The second moment condition leads to:

$$E[(X^B{}_t)^2] = E[(X_t^{\,Y})^2] \Leftrightarrow \sigma^2 t = (2c/k^2)t \qquad \text{(a.7)}$$

Thus, if we construct two distinct processes $\{X^{B}{}_{t}, X_{t}^{Y}\}$ with parameters {*g*} and $\{\sigma^2 = 2c/k^2\}$ they will share the first two – and as a result of symmetry the third as well – logarithmic moments.

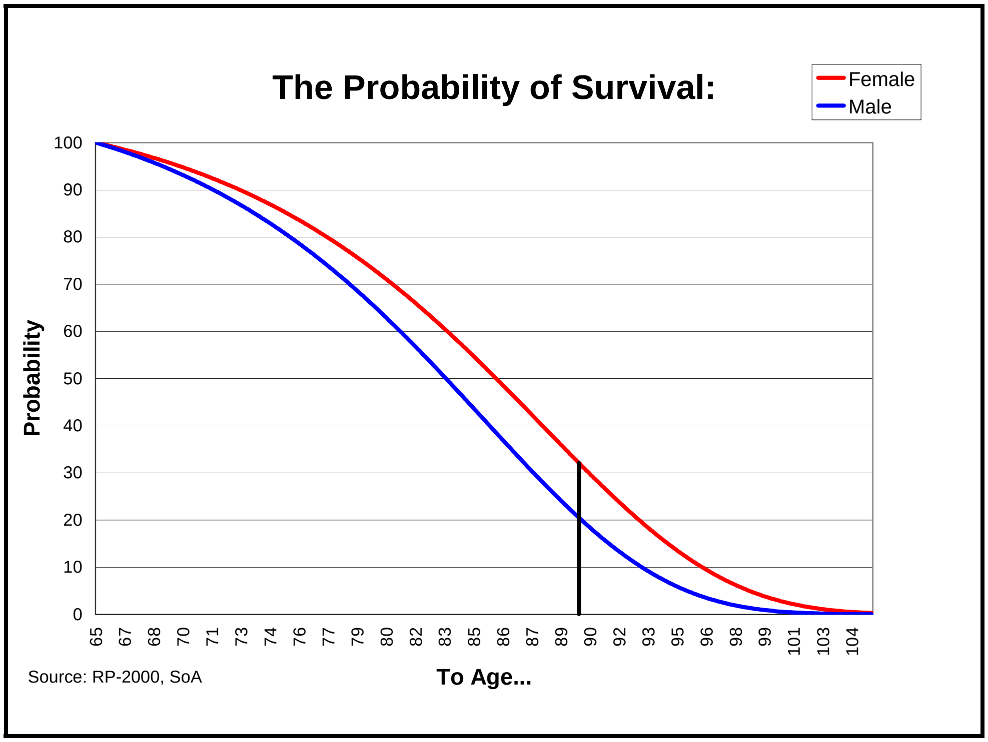
For example, if we set *c = 3* and *k = 12.25*, the second moment of the log-return process is *0.04t*, as per equation (a.7), which then "moment matches" to a geometric Brownian motion with volatility $\{\sigma = 20\%\}$. Along these lines, Table 3 summarizes the results of the simulations. The main insights are as follows. At young ages the ERG approximation – which is based on diffusion processes -- provides very poor results compared to the "true" ruin simulations under a pure jump process. However, at higher ages the accuracy improves and within the spending ranges discussed in the paper, the formula is quite reliable. In sum, the reader should be careful not to read too much into discrepancies between the two numbers given the completely different price process underlying the calculation, but the qualitative results should be noted.

**Table #1**

| Conditional Probability of Survival at Age 65 | | |
|---|---|---|
| **To Age** | **Female** | **Male** |
| **70** | 93.9% | 92.2% |
| **75** | 85.0% | 81.3% |
| **80** | 72.3% | 65.9% |
| **85** | 55.8% | 45.5% |
| **90** | 34.8% | 23.7% |
| **95** | 15.6% | 7.7% |
| **100** | 5.0% | 1.4% |
| **Source:** Society of Actuaries RP-2000 Table (with full projection) | | |

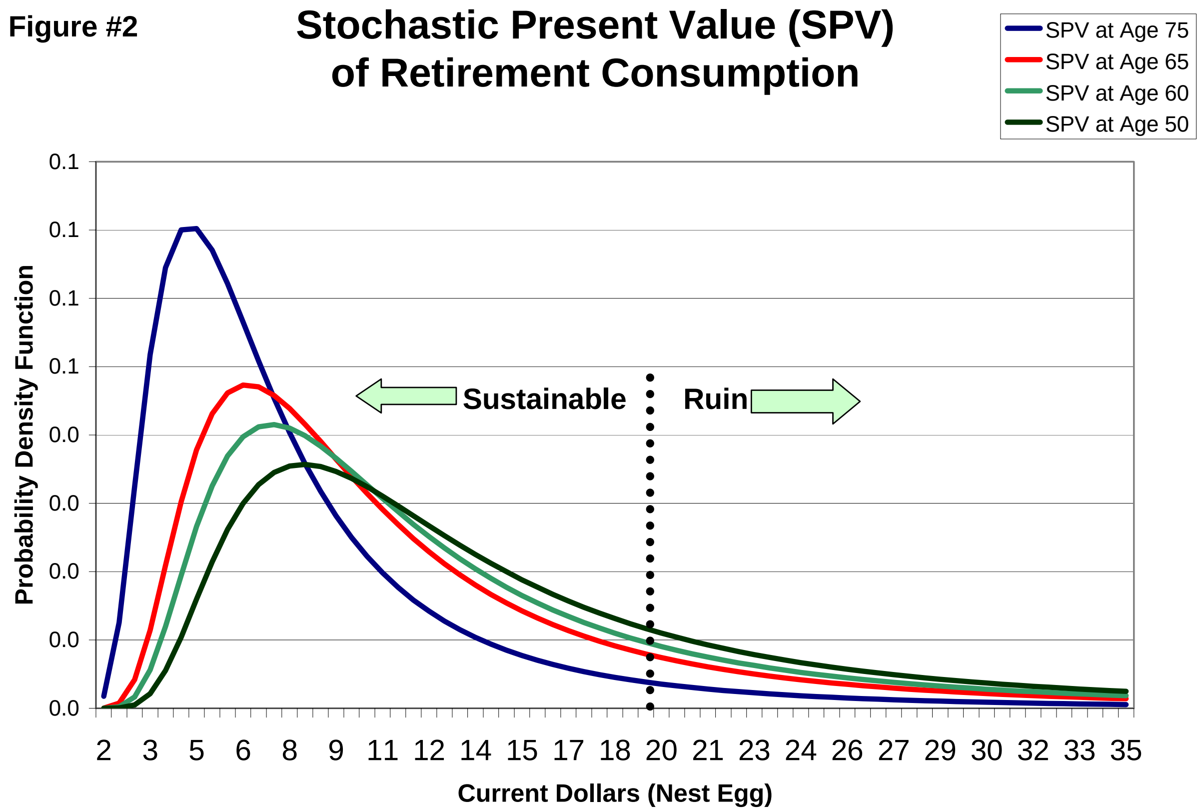
Caption: Table #1 illustrates the randomness of the retiree's investment horizon. A 65 year-old might only survive for 10 years, or might live for an additional 30 years. This uncertainty should be incorporated in any financial advice regarding spending rates.

**Figure #1**



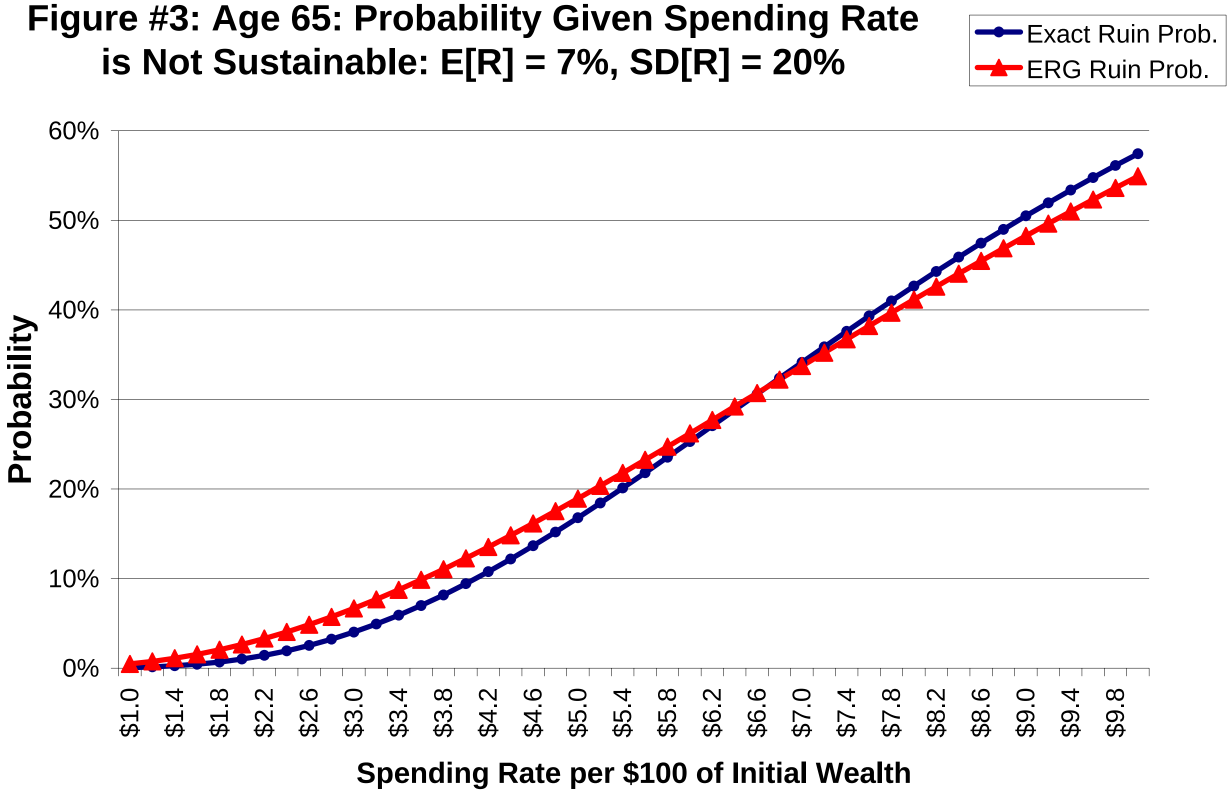

Caption: The figure provides a graphical illustration of the numbers in Table #1. The conditional probability of living to any given age declines – almost exponentially – with time and eventually reaches zero.

**Figure #2**




Caption: When the time horizon and the investment return in a classical cash-flow discounting calculation are unknown, the present value becomes a random variable with the following general shape.

**Figure #3: Age 65: Probability Given Spending Rate is Not Sustainable: E[R] = 7%, SD[R] = 20%**




Caption: How good is the approximation? The exact ruin probability is computed using the RP2000 mortality table and then compared to the ERG approximation which is based on assuming an exponential future lifetime with the same median lifetime as the mortality table.

## Table #2a

**What is the probability you will run out of money -- i.e. spending is not sustainable?**
**Random lifespan & random returns under a fixed spending rate:**

| Retirement Age | Median Age-at-Death | Mortality λ | | $ 2.0 | $ 3.0 | $ 4.0 | $ 5.0 | $ 6.0 | $ 7.0 | $ 8.0 | $ 9.0 | $ 10.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| N.A. | infinity | 0.00% | Approx. | 15.1% | 30.0% | 45.1% | 58.4% | 69.4% | 77.9% | 84.4% | 89.1% | 92.5% |
| | | | Exact. | 15.1% | 30.0% | 45.1% | 58.4% | 69.4% | 77.9% | 84.4% | 89.1% | 92.5% |
| | | | Diff. | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| 50 | 78.1 | 2.47% | Approx. | 4.27% | 10.27% | 18.0% | 26.8% | 35.8% | 44.6% | 52.8% | 60.3% | 66.9% |
| | | | Exact. | 3.04% | 9.10% | 17.8% | 27.7% | 37.8% | 47.2% | 55.5% | 62.6% | 68.5% |
| | | | Diff. | 1.2% | 1.2% | 0.3% | -0.9% | -2.0% | -2.6% | -2.7% | -2.3% | -1.6% |
| 55 | 83.0 | 2.48% | Approx. | 4.26% | 10.23% | 18.0% | 26.7% | 35.7% | 44.5% | 52.7% | 60.2% | 66.8% |
| | | | Exact. | 2.83% | 8.95% | 18.0% | 28.7% | 39.6% | 49.9% | 59.0% | 66.7% | 73.0% |
| | | | Diff. | 1.4% | 1.3% | 0.0% | -2.0% | -3.9% | -5.4% | -6.3% | -6.5% | -6.3% |
| 60 | 83.4 | 2.96% | Approx. | 3.48% | 8.54% | 15.3% | 23.1% | 31.4% | 39.7% | 47.6% | 55.0% | 61.7% |
| | | | Exact. | 1.82% | 6.36% | 13.7% | 22.9% | 32.9% | 42.7% | 51.7% | 59.6% | 66.4% |
| | | | Diff. | 1.7% | 2.2% | 1.6% | 0.2% | -1.5% | -3.0% | -4.1% | -4.6% | -4.6% |
| 65 | 83.9 | 3.67% | Approx. | 2.64% | 6.68% | 12.27% | 18.9% | 26.2% | 33.7% | 41.1% | 48.3% | 54.9% |
| | | | Exact. | 1.02% | 4.03% | 9.43% | 16.8% | 25.3% | 34.1% | 42.7% | 50.5% | 57.4% |
| | | | Diff. | 1.6% | 2.7% | 2.8% | 2.1% | 0.9% | -0.4% | -1.5% | -2.2% | -2.5% |
| 70 | 84.6 | 4.75% | Approx. | 1.81% | 4.73% | 8.95% | 14.2% | 20.1% | 26.5% | 33.0% | 39.5% | 45.8% |
| | | | Exact. | 0.48% | 2.20% | 5.71% | 11.0% | 17.6% | 24.9% | 32.4% | 39.6% | 46.4% |
| | | | Diff. | 1.3% | 2.5% | 3.2% | 3.2% | 2.6% | 1.6% | 0.6% | -0.1% | -0.6% |
| 75 | 85.7 | 6.48% | Approx. | 1.07% | 2.90% | 5.69% | 9.32% | 13.6% | 18.5% | 23.6% | 29.0% | 34.4% |
| | | | Exact. | 0.18% | 0.98% | 2.89% | 6.10% | 10.5% | 15.8% | 21.7% | 27.7% | 33.7% |
| | | | Diff. | 0.9% | 1.9% | 2.8% | 3.2% | 3.1% | 2.6% | 1.9% | 1.2% | 0.7% |
| 80 | 87.4 | 9.37% | Approx. | 0.52% | 1.47% | 3.00% | 5.10% | 7.71% | 10.8% | 14.2% | 18.0% | 21.9% |
| | | | Exact. | 0.05% | 0.34% | 1.16% | 2.76% | 5.20% | 8.43% | 12.3% | 16.6% | 21.1% |
| | | | Diff. | 0.5% | 1.1% | 1.8% | 2.3% | 2.5% | 2.3% | 1.9% | 1.4% | 0.8% |

| | |
|---|---|
| Return: | 7.0% |
| Volatility: | 20.0% |

Caption: The table compares the results of the ERG approximation presented in the paper against the exact results using the correct mortality table. For the exact solution I used the Unisex (non-projected) RP2000 mortality table from the Society of Actuaries. For the approximate solution I used an exponential future lifetime assumption matched to the RP2000 median age-at-death.

**Table #2b**

**What is the probability you will run out of money -- i.e. spending is not sustainable?**
**Random lifespan & random returns under a fixed spending rate:**
**ERG Approximation assuming: High (20%) volatility & low (5%) return portfolio**

| Age | MED[T] | λ | $ 2.00 | $ 3.00 | $ 4.00 | $ 5.00 | $ 6.00 | $ 7.00 | $ 8.00 | $ 9.00 | $ 10.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| N.A. | infinity | 0.00% | 42.8% | 60.8% | 73.9% | 82.8% | 88.8% | 92.8% | 95.4% | 97.1% | 98.1% |
| 50 | 28.1 | 2.47% | 11.5% | 21.9% | 32.9% | 43.5% | 53.2% | 61.7% | 68.9% | 75.0% | 80.0% |
| 55 | 28.0 | 2.48% | 11.5% | 21.8% | 32.8% | 43.4% | 53.1% | 61.5% | 68.8% | 74.9% | 80.0% |
| 60 | 23.4 | 2.96% | 9.1% | 18.1% | 28.0% | 37.9% | 47.2% | 55.7% | 63.1% | 69.6% | 75.1% |
| 65 | 18.9 | 3.67% | 6.7% | 13.9% | 22.3% | 31.1% | 39.8% | 48.0% | 55.5% | 62.2% | 68.1% |
| 70 | 14.6 | 4.75% | 4.4% | 9.6% | 16.1% | 23.3% | 30.8% | 38.1% | 45.2% | 51.9% | 58.0% |
| 75 | 10.7 | 6.48% | 2.4% | 5.7% | 10.0% | 15.1% | 20.8% | 26.7% | 32.7% | 38.7% | 44.4% |
| 80 | 7.4 | 9.37% | 1.1% | 2.7% | 5.0% | 8.0% | 11.5% | 15.5% | 19.7% | 24.1% | 28.6% |

| | |
|---|---|
| **Return:** | 5.0% |
| **Volatility:** | 20.0% |

Caption: The table assumes a lower (5%) investment return and should be contrasted with the results in Table #2a where the expected return was 7%.

## Table #2c

**What is the probability you will run out of money -- i.e. spending is not sustainable?**
**Random lifespan & random returns under a fixed spending rate:**
**ERG Approximation assuming: Low (10%) volatility & low (5%) return portfolio**

| Age | MED[T] | λ | $ 2.00 | $ 3.00 | $ 4.00 | $ 5.00 | $ 6.00 | $ 7.00 | $ 8.00 | $ 9.00 | $ 10.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| N.A. | infinity | 0.00% | 2.1% | 15.3% | 40.7% | 66.7% | 84.5% | 93.8% | 97.8% | 99.3% | 99.8% |
| 50 | 28.1 | 2.47% | 1.0% | 4.4% | 10.9% | 20.2% | 31.3% | 42.9% | 54.0% | 64.0% | 72.5% |
| 55 | 28.0 | 2.48% | 1.0% | 4.3% | 10.8% | 20.1% | 31.2% | 42.8% | 53.9% | 63.9% | 72.4% |
| 60 | 23.4 | 2.96% | 0.9% | 3.6% | 9.0% | 16.8% | 26.3% | 36.7% | 47.1% | 56.8% | 65.5% |
| 65 | 18.9 | 3.67% | 0.7% | 2.8% | 7.0% | 13.2% | 21.0% | 29.8% | 38.9% | 47.9% | 56.4% |
| 70 | 14.6 | 4.75% | 0.5% | 2.0% | 5.0% | 9.5% | 15.3% | 22.1% | 29.6% | 37.3% | 45.0% |
| 75 | 10.7 | 6.48% | 0.3% | 1.3% | 3.1% | 6.0% | 9.9% | 14.6% | 19.9% | 25.8% | 31.9% |
| 80 | 7.4 | 9.37% | 0.2% | 0.7% | 1.7% | 3.2% | 5.4% | 8.1% | 11.3% | 15.0% | 19.1% |

| | |
|---|---|
| **Return:** | 5.0% |
| **Volatility:** | 10.0% |

Caption: The table assumes a lower (5%) return and (10%) volatility and should be contrasted with the results in Table #2b and #2a.

**Table 3**

Assume Portfolio Returns are Generated by a pure Jump Process,
with AM return of 7% standard deviation of 20% and GM return = 5%
What is the Probability the given Spending Rate is Not Sustainable?

| **Age:** | | **$3.50** | **$4.00** | **$4.50** | **$ 5.00** |
|---|---|---|---|---|---|
| **65** | *Simulation:* | 3.86% | 12.60% | 24.06% | 38.92% |
| | *ERG Apporox.* | 9.31% | 12.27% | 15.49% | 18.93% |
| **70** | *Simulation:* | 0.28% | 2.22% | 7.36% | 14.74% |
| | *ERG Approx.* | 6.69% | 8.95% | 11.46% | 14.18% |

Caption: The algorithm moment-matches the pure jump process to a lognormal density with the same (logarithmic) mean and standard deviation. I then simulated lifetime ruin probability using the jump process and compared with the ruin probability using the ERG approximation.