A Gentle Introduction to the Calculus of Retirement Income: What is Your Retirement RisQuotient?

June 01, 2007

By: *Moshe A. Milevsky, Ph.D.*[1]
Associate Professor of Finance
York University, Toronto, Canada

ABSTRACT:

A little over a year ago, on January 1st, 2006 the first American baby boomer turned 60. These birthdays are expected to continue at the rate of one per 7-10 seconds over the next 20 years. In anticipation of this demographic wave the financial services industry is bracing for the retirement income revolution. And, one of the critical issues is how to build a portfolio that will provide a sustainable income flow over the uncertain length and cost of the human lifecycle. Indeed, a number of recent articles have gained notoriety by advocating spending rates in the 4% to 6% vicinity as being sustainable for portfolios that contain 70% to 90% equity exposure.

But, prudent risk management involves more than just controlled consumption and in this article I deliberately avoid advocating a particular spending rate. Instead I provide an overview of the analytic relationship between the key risk variables that determine sustainability. I bring these ingredients together by linking investment characteristics, spending rates and longevity risk, to an analytic Retirement RisQuotient. And, while statistical formulas will never capture the complex nuances of retirement reality, there are a variety of intuitive insights that can be gleaned from this summary number. Moreover, this calculus illustrates how products with longevity insurance (i.e. life annuities) and downside protection (i.e. put options) can increase the sustainability by reducing the Retirement RisQuotient.

[1] Moshe A. Milevsky is also the Executive Director of The IFID Centre, 222 College Street, Toronto, Ontario, Canada, M5T 3J1. He can be reached via email at milevsky@yorku.ca, or by phone at 416-348-9710 x 3010. He would like to thank Chris Robinson and Tom Salisbury for many helpful discussions and Anna Abaimova for extensive research assistance.

SECTION #1:
INTRODUCTION AND MOTIVATION: BOOMER WANTS INCOME

The last few years have seen intense research around the topic of retirement income planning. And, one of the most intensely studied issues is how to build a portfolio that will provide a sustainable income flow over the uncertain length and cost of the human lifecycle. Not surprisingly, a number of recent articles have argued that one of the key components to a sustainable retirement is to simply avoid spending too much. A variety of authors have advocated spending rates in the 4% to 6% vicinity as being sustainable for portfolios that contain 70% to 90% equity exposure.

For example, Bengen (1994), Ho, Milevsky, and Robinson (1994), Cooley, Hubbard, and Walz (1998, 2003), Pye (2000), Milevsky (2001), Ameriks, Veres, and Warshawsky (2003), and Guyton (2004) have all created financial experiments incorporating historical, simulated, and scrambled returns to quantify the sustainability of various *ad hoc* spending policies and consumption rates for retired individuals. The problem with such studies is that (1) these simulations are often difficult to replicate by financial practitioners, (2) many conduct only a minimal number of simulations and (3) most provide little pedagogical intuition on the financial trade-off between retirement risk and return. More importantly, they focus too much attention on the relationship between spending habits and the composition of the investment portfolio, yet they neglect the many other factors that determine retirement income sustainability. For example, important questions such as: "What is the role of variable payout annuities and their guaranteed riders in a sustainable portfolio?" are not addressed in any of these studies.

Therefore, to differentiate this article from the recent debate around prudent spending habits, I deliberately avoid advocating a preferred portfolio withdrawal rate. Instead, I provide an overview of the analytical relationship between the key risk variables that determine sustainability. I bring these ingredients together by linking portfolio parameters, spending rates and longevity risk, to an analytic probability of retirement ruin. More specifically, I identify a number of key variables that will determine a retiree's Retirement RisQuotient

And, while statistical formulas will never capture the complex nuances of retirement reality, there are a variety of intuitive insights that can be gleaned from this summary number. Moreover, this calculus illustrates how products with longevity insurance (i.e. life annuities) and downside protection (i.e. the put options embedded within variable annuities) can increase the sustainability of retirement.

The remainder of this article is organized as follows. In the next section (#2) I present the basic formula that helps us measure a retiree's RisQuotient. The formula uses a function that is easily available in Microsoft Excel. The following section (#3) provides a short-cut method for arriving at the RisQuotient using two summary variables. The subsequent section (#4) discusses some caveats and warnings about the underlying assumptions while the next section (#5) illustrates how this risk measurement system can be extended to include life annuities and various downside protected portfolio strategies, many of which are embedded within variable annuities. The final section (#6) concludes the article.

SECTION #2:
THE MAIN FORMULA: PROBABILITY OF RETIREMENT RUIN

If a retiree is invested in a standard (balanced) portfolio and plans on withdrawing a fixed inflation-adjusted amount every year during retirement, he or she obviously faces the probability their portfolio will be exhausted while still alive. Under basic portfolio assumptions, the formula for this *probability of retirement ruin* – which is the probability that a fixed spending plan will deplete a retirement nest egg prior to the end of the lifecycle – can be expressed as follows:

$$\text{RetirementRuin} := \text{Wealth is Zero \& You are Alive} = \frac{\int_0^{S/\beta} y^{(\alpha-1)} e^{-y} dy}{\int_0^{\infty} y^{(\alpha-1)} e^{-y} dy}, \qquad \text{(eq.1)}$$

$$\alpha = \frac{2(\mu + 2\lambda)}{\sigma^2 + \lambda} - 1, \quad \beta = \frac{\sigma^2 + \lambda}{2}$$

where $\{\mu\}$ denotes the retiree's portfolio expected rate of return, $\{\sigma\}$ denotes the volatility or uncertainty surrounding this projected return, $\{\lambda\}$ denotes the mortality rate of the retiree and $\{S\}$ denotes the inflation-adjusted spending rate as a percentage of the initial portfolio value.

For those readers who disliked, avoided or failed calculus in college, this formula might seem a tad overwhelming. But, thanks to modern technology, Microsoft Excel now does the heavy lifting for you and this formula can be implemented with ease using a simple pull-down menu. The easy way of using formula (eq.1) is via:

$$\text{RetirementRuin} = \text{GammaDist}(\text{S}/\beta, \alpha, 1, \text{TRUE}) , \qquad \text{(eq.2)}$$

where the variables $\{\alpha, \beta\}$ are defined by (eq.1) as explicit functions of the input variable $\{\mu, \sigma, \lambda, S\}$, all of which will be explained in greater detail, in a moment.

First, let me break (eq.1) into bite-size pieces and translate the Greek into English. The formula depends on four input variable $\{\mu, \sigma, \lambda, S\}$, which get mapped into the two intermediate variables $\{\alpha, \beta\}$, which then get used in the actual formula. The *retirement ruin* probability is the ratio of two similar-looking integrals. The integral in the numerator of equation (eq.1) ranges from the lower bound of zero to the upper bound of a *beta-adjusted spending* rate. The integral in the denominator ranges from a lower bound of zero to an upper bound of infinity. The integrand – i.e. the expression $\{y^{(\alpha-1)} e^{-y}\}$ which appears in both the numerator and denominator -- is the same. In fact, this particular function is called the Gamma density function, and is well known to statisticians.

One of the first insights from (eq.1) is that when the retirement spending rate *S* is very (very) large and therefore the upper bound of integration goes to infinity, the numerator and denominator get very close to each other and the retirement ruin probability approaches 100%. Likewise, when the spending rate approach zero and thus $\{S/\beta \rightarrow 0\}$ is formula (eq.1), the integral in the numerator approach zero as well, and the retirement ruin probability is zero.

Beyond that rather trivial insight, here is a numerical example to help develop a better understanding of how to use the formula in general. Imagine that you have a client who is a 65 year-old female who has just retired with a nest egg of $500,000 which is sitting inside a tax-sheltered plan. Any and all withdrawals from this plan will be taxed at the retiree's marginal tax rate, which is why I will not distinguish between the various forms of income such as taxable bonds, tax-efficient equity, etc.

According to mortality tables used by pension actuaries, your 65 year-old clients each has (approximately) a 94% chance of surviving for 5 more years to age 70, a 56% chance of surviving for 20 more years to age 85, and a 16% of surviving for 30 more years to age 95. Their retirement horizon is random and they obviously face the <u>longevity risk</u> of outliving their nest eggs if they live longer than anticipated. It is hard to overstate the importance of incorporating longevity risk when preparing a retirement plan. The many research studies (and software algorithms) that focus deterministically on 20, 25 or 30 years of retirement are ignoring the most vexing risk of all.

Either way, your 65 year old (healthy) clients' <u>median remaining lifespan</u> (MRL) is approximately 23 more years, which means that there is 50% chance they will live beyond age 88. From a mathematical point of view, if there is a 50% chance of living for 23 more years, the <u>exponential lifetime</u> assumption $\{e^{-23\lambda} = 0.5\}$ leads to a mortality rate of $\{\lambda = \ln[2]/23 = 0.0301\}$. The average rate of death is approximately 3% per year. This number is one of the four input variables in (eq.1).

Let me further assume that one of these 65 year-olds would like to withdraw \$40,000 per year, pre-tax in inflation-adjusted terms, during the course of his retirement. This represents $\{S = 40/500 = 0.08\}$ or 8% of his initial nest egg and is another one of the four input variables in the retirement ruin formula.

Now, let me state-up front that it is unrealistic to assume your client will be mechanically adhering to a withdrawal rule of precisely \$40,000 per year in inflation-adjusted terms. He obviously has the flexibility to modify his plan by spending more or less in any given year. Most likely if the portfolio performs poorly he will scale back his spending plans and *vice versa* if he achieves above average investment performance. However, the point of this exercise is to ascertain whether \$40,000 per year withdrawals are sustainable over a random retirement horizon.

Finally, assume that your client's investment portfolio is allocated to a mix of balanced mutual funds (or ETFs, SMAs, etc) and you estimate that this portfolio will earn $\{\mu = 0.075\}$, which is an arithmetic average of 7.5% in any given year, with a standard deviation or volatility of $\{\sigma = 0.18\}$, which is 18% per year, both in inflation-adjusted terms. These numbers are consistent with the well-known and widely-cited numbers by Ibbotson Associates (2004). In some sense, the precise magnitude of these numbers is less important than the general awareness that they can fluctuate and hence are yet another source of retirement risk.

To those who are unfamiliar with investment means and volatilities, what these parameters imply is that in any given year the range of investment outcomes is between minus 10.5% and plus 25.5% two times out of three, which is one standard deviation away from the mean. If you want nineteen times out of twenty confidence, the range is between minus 28.5% and plus 43.5%, which is obviously much wider. Note, also, that all of these numbers are expressed as a continuously compounded annual percentage rate. Either way, the two numbers 0.075 and 0.18 are the last of the four ingredients we need to use the retirement ruin equation.

The next step is to compute the *retirement alpha:* $\{\alpha = \frac{2\mu + 4\lambda}{\sigma^2 + \lambda} - 1 = 3.326\}$ and the *beta-adjusted spending* rate: $\{\frac{S}{\beta} = \frac{2S}{\sigma^2 + \lambda} = 2.558\}$, as per the instructions in (eq.1). Finally, plugging these values into Microsoft Excel's function as per (eq.2), *GammaDist(2.558,3.326,1,TRUE)* results in the number 39.3%, which is the probability of retirement ruin. Intuitively, a higher retirement alpha will result in a lower risk of retirement ruin, while a higher beta-adjusted spending rate will lead to a higher retirement ruin, a.k.a. RisQuotient.

Now, for example, if this retiree was planning on spending only $35,000 per year of retirement (in inflation adjusted terms), the spending rate would fall to 7% of the initial $500,000 and the beta-adjusted spending rate would be reduced to 2.239 instead of the earlier 2.558 value. It is therefore no surprise that when this value is plugged into the same function *GammaDist(2.239,3.326,1,TRUE)* the probability of retirement ruin is reduced to 31.3% from 39.3%. A lower beta-adjusted level of spending results in a lower ruin probability. Likewise, if instead of assuming that the portfolio will earn 7.5% in any given year, I am more optimistic and use $\{\mu = 0.09\}$, which is 1.5% per annum above the earlier number, the retirement alpha variable becomes 3.806 instead of the earlier 3.326 value. And, under the earlier $40,000 spending rate and beta-adjusted spending rate of 2.558, we arrive at a retirement ruin probability, using the same formula *GammaDist(2.558,3.806,1,TRUE)* of 29.1% instead of 39.3%. This is an improvement – i.e. a reduction of risk -- of close to ten percentage points. Once again, a higher retirement alpha value will reduce the ruin probability.

Tables #1 - #4 provide a range of numbers based on (eq.1) assuming differing spending rates, mortality rates and portfolio characteristics. The four tables correspond with four extremes. Table #1 represents a relatively high spending rate of 8% of a typical retiree with a 23-year median remaining lifespan. Table #2 illustrates the results for the same 8% spending rate, but for a much younger (early) retiree facing a median remaining lifespan of 35 years. Table #3 and #4 examine the same mortality rates but under a much lower spending rate of 4% of the initial nest egg. Recall once again that all of these rates are adjusted for inflation.

Table #1, #2, #3 and #4 placed here

Thus, for example, a portfolio that is expected to earn 5% per annum with a volatility of 10% will result in a 46.4% ruin probability under a 8% spending rate and 23-year median remaining lifespan, but only an 8.8.% probability of ruin under a 4% spending rate. Halving the spending rate reduces the ruin by over 38 percentage points. It is often stated that spending money in retirement is akin to creating your own pseudo "bear market" since each year the withdrawal process reduces portfolio growth by the spending rate S. Notice that, consistent with this heuristic, increasing S from 4% to 8% under a 7% expected investment return and a 20% volatility, has the same impact as reducing the expected return from 7% to 3% under an 8% spending rate.

The important take-away is as follows. I am not advocating a particular spending/withdrawal rate or equity/bond allocation as being optimal or prudent. Rather, the answers to all these important questions are intertwined and depend on the level of risk aversion (or tolerance) of the retiree, compared against the RisQuotient of the retirement plan.

The point is that I can defend spending rates as high as 8% to 10% if the forward-looking volatility is low enough and the forward-looking expected return is high enough. Thus, the retirement income debate should be phrased in terms of the anticipated and forward-looking equity risk premium as opposed to the historical performance of these asset classes. The formula (eq.1) provides this link.

Figure #1 provides a graphical illustration of the retirement ruin probabilities under the same 7.5% expected return and 18% volatility assumption, for a variety of spending rates and at various retirement ages. The differing retirement ages are captured by changing the median remaining lifespan (MRL). At the extreme left of the figure there is a 50% chance of living for 5 more years and at the extreme right there is a 50% chance of living for 40 more years. Remember that the MRL is used to compute the mortality rate $\{\lambda\}$ which is as the heart of our formula (eq.1).

Figure #1 placed here

As one would expect, the greater the spending rate – which ranges from a low of 2% to a high of 10% -- the higher the probability of retirement ruin. Likewise, the longer the median remaining lifespan, the greater is the probability of ruin. Notice that when you are planning on spending 10% of your initial nest egg each and every year of retirement, the probability of ruin ranges from a low of about 10% to a high of 69% depending on how old you are when you start spending. In contrast, when you are planning on spending only 2% of your nest egg during retirement, the ruin probability ranges from a minimal 0.1% to 2.7% depending on age.

Another possible way of interpreting or thinking about the median remaining lifespan (MRL) is to treat this number as the 50% mark for a couple's retirement plan. If the couple expects at least one member to be alive for at least 30 years, then you should focus on numbers to the right of 30 on the x-axis. Those who can tolerate living on the 30% ruin fault line can obviously spend much more than those who prefer a southern climate in the 10% ruin region.

SECTION #3:
ON THE BACK OF AN ENVELOPE: NO SOFTWARE REQUIRED

It is possible to use and apply the main (eq.1) or (eq.2) without having access to Microsoft Excel, or any other statistical software package that does calculus. In fact, using the two key variables of *retirement alpha* $\{\alpha\}$ and the *beta-adjusted spending* rate $\{S/\beta\}$, you can calculate the retirement ruin probability on the back of a (large) envelope. Table #5 provides the main exhibit.

Table #5 Placed Here

The columns of table #5 represent increasing levels of beta-adjusted spending, while the rows capture decreasing levels of retirement alpha. From any given cell in the table, moving down and/or to the right will increase the probability of retirement ruin. Quite naturally, if you are spending more or generating a smaller retirement alpha, the probability of ruin and RisQuotient is higher. Within this table the numbers range from a low of almost zero to a high of 93%. Ideally you want to position yourself in the upper left-hand corner of this two dimensional risk map.

For example, using the 8% spending rate numbers from the previous section, when the inflation-adjusted portfolio mean return was 7.5%, the return volatility was 18% and mortality rate was 3.01%, the value of the retirement alpha was 3.326 and the beta-adjusted spending was 2.558. This led to a probability of retirement ruin of 39.3%. Now, if we examine table #5, the closest value for $\{\alpha\}$ is 3.3 and the closest value for $\{S/\beta\}$ is somewhere between 2.45 and 2.6, with equal distance. This leads to a probability of 37% to 41%, which closely approaches the true 39.3% probability. Likewise, if the spending is reduced from 8% of the initial nest egg to 4% of the nest egg, the retirement alpha value remains unchanged but the beta-adjusted spending rate declines to 1.279 from 2.558. The exact retirement ruin probability is now 9.5% according to formula (eq.1). In table #5 when the beta adjusted spending rate falls between 1.25 and 1.4, the ruin probability is between 9.3% and 12.1%.

If you decide to use such a table instead of the precise formula note that occasionally you might encounter portfolio return variables $\{\mu,\sigma\}$ and/or mortality rates $\{\lambda\}$ and/or inflation-adjusted spending rates $\{S\}$ that will induce $\{\alpha, S/\beta\}$ values lying beyond the ranges listed in table #5. In this case, the direction via which you are leaving the table should provide you with a general picture of sustainability. If, for example, your computed retirement alpha value is very low (<1), this is bad news. The ruin probability will be high. And, so is a very large (>3) level of your beta-adjusted spending rate. Of course, if you want a more precise answer than what the table's heuristics can provide, then fire-up your Excel.

In sum, the way to use this table is in two stages. First you must input the main variables $\{\mu, \sigma, \lambda, S\}$ and convert those into the key variables $\{\alpha, S/\beta\}$. Then, you take these two summary numbers to table #5 and look up the retirement ruin probability based on rows and columns. Obviously, this exercise will not produce the precise number generated by formula (eq.1) and (eq.2) since we are approximating the row and column variable to within two digits, but you can see the results are quite close. In fact, given that this entire exercise hinges on a number of embedded assumptions that I will describe in the next section, I would avoid emphasizing numerical results beyond the first two digits of table #5.

Using this approach, one can envision a simple questionnaire that clients (users) complete to measure their Retirement RisQuotient. Retirees would be asked about their gender, age (family history, health) information to arrive at an estimate of their median remaining lifespan (MRL) and their implied mortality rate $\{\lambda\}$. The same user would then specify their inflation-adjusted retirement income spending rate net of any defined benefit (DB) pension income, which would lead to an estimate for $\{S\}$. Finally, they would be asked about the specific composition of their investment portfolio that would lead to estimates of $\{\mu, \sigma\}$. All of these together would be fed-into a formula or table #5 that would generate a Retirement RisQuotient. The higher this number the lower the sustainability of the retirement plan.

SECTION #4
CAVEATS AND WARNINGS FOR THE QUANTS: KNOW THE LIMITS

For those readers who want to know where this formula comes from, or how accurate it is relative to extensive Monte Carlo simulations, I recommend they consult the book by Milevsky (2006) or the paper by Milevsky and Robinson (2005). Both references contain much greater mathematical detail relative to what can be described in a brief article such as this. In a nutshell, however, there are a number of important assumptions upon which equation (eq.1) is based. Allow me to spell them out.

First, I am assuming that the underlying investment portfolio obeys a geometric Brownian motion (a.k.a. GBM). This means that in any given year the one-plus-investment return is lognormally distributed and the logarithm of the one-plus-investment return (a.k.a. the continuously compounded return) is normally distributed. And, while this embedded assumption has been used by thousands of practitioners ever since the pioneering days of Markowitz and Merton it has been questioned by thousands of academics as well. In GBM's defense, it seems the brunt of the criticism has been leveled at the inability of the normal distribution to capture the rare and infrequent tails in market returns over short horizons, as opposed to the (very) long term which would be relevant for the sustainability of retirement portfolios. Thus, I feel comfortable using the normal distribution within the calculus of retirement income, although I tend to use a portfolio volatility estimate $\{\sigma\}$ that is slightly higher than historical estimates to take account of the above-mentioned model risk.

Second, the model is predicated on the uncertain length of human life being exponentially distributed. This implies that the mortality rate is constant over time and that the probability of living for $\{T\}$ more years is precisely $\{e^{-\lambda T}\}$, where $\{\lambda\}$ denotes the instantaneous mortality rate. Indeed, many insurance actuaries will recoil in horror from this assumption knowing that the only organism on earth with an exponential remaining lifespan is a lobster! Human mortality rates increase with age and certainly do not remain constant over time. Yet, interestingly enough, when (eq.1) is calibrated to the actuarially-correct median lifespan the results are remarkably accurate when compared against the true ruin probability under the complete mortality rates.

Third, even under an exponential remaining lifespan and lognormally distributed investment returns, (eq.1) is an approximation based on moment matching techniques. In other words, the ruin probability is accurate in the limit and only when $\{\lambda \to 0\}$ does the formula converge to the truth. Once again, this approximation has been stresstested quite extensively in a number of other papers – see Milevsky and Robinson (2005) for example – and the results indicate errors that are less than 5% in the most extreme cases.

Finally, not withstanding the above mentioned caveats there are a number of important limits to the range of (eq.1). For example, the retirement alpha value must be greater than zero, or the integral "blows up" and the formula produces errors. This places restrictions on the *geometric mean* investment return $\{\mu - 0.5\sigma^2\}$ relative to the mortality rate $\{\lambda\}$. Specifically, we must obey the relationship $\{\mu - 0.5\sigma^2 + 1.5\lambda > 0\}$ in order to satisfy $\{\alpha > 0\}$. Thus, if you want to use the formula for any retirement age and any mortality rate, a sufficient condition is that the geometric mean investment return is positive. Notice the critical role of the geometric mean investment return $\{\mu - 0.5\sigma^2\}$ relative to the arithmetic mean investment return $\{\mu\}$, in determining the range of sustainable spending rates. Investment volatility matters!

SECTION #5:
ENGINEERING SUSTAINABLE INCOME: THE ROLE OF PRODUCT ALLOCATION

In addition to the traditional menu of asset classes that are available to retirees, there are two important product classes that can further reduce the probability of retirement ruin and increase the sustainability of a retirement plan. The first product class is a life annuity (a.k.a. payout or income annuity) and the second product class consists of put options used to protect the portfolio. And, while many retirees and their advisors might shun exchange traded options, the closest retail substitute is a variable annuity – manufactured by insurance companies -- that offer guaranteed withdrawal benefits. These products effectively create downside protection in the critical early years of retirement. I'll explain the technical merits of each product class in a moment. My main point is to stress the importance of expanding the classical opportunity set from asset classes to product classes.

Buying a life annuity -- which provides lifetime income – can greatly improve sustainability. This point was made by Ameriks, Veres and Warshawsky (2001) using simulation arguments, or by Chen and Milevsky (2003) using utility-based arguments, the same insight can also be extracted from (eq.1) in a number of ways. First, by reducing the amount of income that has to be withdrawn from the portfolio, the retiree is exposed to a lower level of longevity risk. This reduces the probability of retirement ruin. More importantly, annuitizing a percentage of one's nest egg generates a mortality subsidy, which increases the portfolio's investment return. This concept has been explained in Milevsky (2005), but can be illustrated using a simple story.

Imagine that a large group of retirees of exactly the same age – and each subject to the same mortality rate $\{\lambda\}$ – pool their retirement nest egg into one large portfolio, whose value I denote by the symbol $\{W_t\}$. The members of the pension portfolio invest their collective wealth $\{W_t\}$ in a mutual fund that is expected to earn $\{\mu\}$ in any given year. Every day the members of this pension fund, who are still alive, withdraw the exact same (*S/365)W*, stated as a percentage of the original nest egg. However, those who do not survive, forfeit their retirement assets to the group. In other words, these funds are not bequeathed to an estate or beneficiary. Now, since this pool of money is being augmented by the assets of the deceased, the money will last longer compared to the situation in which each retiree managed their own portfolio independently. In fact, over a short period of time the portfolio will increase by $\{W(\mu+\lambda)\}$, which is the sum of the expected investment return and the mortality rate.

From a practical perspective, this implies that we can use the exact same formula (eq.1) to compute the probability of retirement ruin, except that we substitute the value of $\{\mu+\lambda\}$ instead of $\{\mu\}$. Clearly this will reduce the risk and increase the sustainability of the portfolio. And, although real-world annuities have additional features that are not quite captured by this framework, the underlying principle is exactly the same.

Table #6 placed here.

Table #6 provides some numerical indication of how annuitization will reduce the probability of retirement ruin, a.k.a. the RisQuotient. For example, if a 65 year-old retiree plans on spending 8% (in real terms) and is anticipating a median age at death of 83.9, then the implied mortality rate is $\{\lambda = \ln[2]/(83.9-65) = 0.03667\}$. The non-annuitized probability of retirement ruin is 41.15%, under portfolio parameters of 7% expected return and 20% volatility. However, if this person completely annuitizes his nest egg – which is obviously the extreme case – the instantaneous expected return will increase from 7% to 10.3667%, due to the mortality credits. In other words, since retirees are dying and abandoning their wealth at a rate of 3.667% per year, the portfolio will grow by an extra 3.667% per year. If we then compute the retirement ruin probability using the exact same $\{\lambda = 0.03667, \sigma = 0.20, S = 0.08\}$ variables, but using 10.366% instead of 7% for the input variable $\{\mu\}$, we are left with a retirement ruin probability of 20.55%. Hence, life annuities in their most general form will reduce the retirement ruin probability and increase sustainability. And, while I have ignored management and insurance fees – which can easily be incorporated by reducing the magnitude of $\{\mu + \lambda\}$ – the underlying message remains the same.

Just as important as the life annuity product class, downside protection (a.k.a portfolio insurance or put options) also reduces the retirement ruin probability. By downside protection I mean having access to derivative products that limit the range of investment outcomes and thus eliminating the "bad tails". These types of guarantees can be purchased in the open market on option exchanges, or the process can be outsourced to intermediaries such as insurance companies that offer withdrawal benefits on variable annuities. All of these guaranteed minimum withdrawal benefits (GMWBs), guaranteed minimum income benefits (GMIBs) and Guaranteed Minimum Accumulation Benefits (GMABs) effectively reduce the magnitude of the volatility variable $\{\sigma\}$ which in turn reduces the retirement ruin probability *assuming the insurance fees charged for this protection are not too high*.

For example, look again at table #1. In this case a portfolio that is expected to earn 7% with a volatility of 20% will result in a 47.1% ruin probability under an 8% withdrawal rate, for a retiree planning for a median lifespan of 23 years. Note that if the volatility of the portfolio's return can be reduced from 20% to 10%, the probability of retirement ruin will (obviously) be reduced from 47.1% to 28.6%, which is substantial. This reduction in volatility would virtually eliminate the possibility of any large and pleasant (upside) surprises in exchange for avoiding large unpleasant (downside) surprises. And, even if the retiree has to pay an extra (abnormally high) fee of 200 basis points which reduces the expected return from 7% to 5%, the probability of retirement ruin still declines from 47.1% to 46.4%. In sum, the main formula (eq.1) illustrates how two important product classes – life annuities and downside protection impact the probability of retirement ruin, all in one parsimonious framework.

SECTION #6:
CONCLUSION: MANAGE YOUR RISK

Simple mathematical formulas in finance can often seduce their users into a false sense of precision regarding the true behavior of the real world. And, while academics might occasionally fall prey to their beauty, seasoned practitioners know the limits of their relevance. Nevertheless, I believe this article contains a number of qualitative insights that do apply to real people (and not only lobsters!)

First, early retirement generates a higher RisQuotient, all else being equal. Likewise, greater spending rates, lower portfolio returns and higher investment volatility all increase the Retirement RisQuotient. Obviously, these insights will not come as news to the financial planning profession. However, this framework has also provided a rigorous justification for using life annuities and other riders associated with variable annuities in one parsimonious framework, since both these products increase the sustainability of a retirement income portfolio and reduce the retiree's RisQuotient.

REFERENCES:

Ameriks, J., M. Veres, and M. Warshawsky. 2001. “Making Retirement Income Last a Lifetime.” *Journal of Financial Planning*, (December): 60-76

Bengen, W.P. 1994. “Determining Withdrawal Rates Using Historical Data.” *Journal of Financial Planning*, vol. 7, no. 4 (October):171–181.

Chen, P. and M.A. Milevsky. 2003. “Merging Asset Allocation and Longevity Insurance: An Optimal Perspective on Payout Annuities”, *Journal of Financial Planning*: 64-72

Cooley, P.L., C.M. Hubbard, and D.T. Walz. 1998. “Retirement Spending: Choosing a Withdrawal Rate That Is Sustainable.” *Journal of the American Association of Individual Investors*, vol. 20, no. 1 (February):39–47.

———. 2003. “Does International Diversification Increase the Sustainable Withdrawal Rates from Retirement Portfolios?” *Journal of Financial Planning*, (January): 74-80.

Guyton, J.T. 2004. “Decision Rules and Portfolio Management for Retirees: Is the ‘Safe’ Initial Withdrawal Rate Too Safe?” *Journal of Financial Planning* (October):50–60.

Ho, K., M. Milevsky, and C. Robinson. 1994. “How to Avoid Outliving Your Money.” *Canadian Investment Review*, vol. 7, no. 3 (Fall):35–38.

Ibbotson Associates. 2004. Stocks, Bonds, Bills and Inflation: 2004 Yearbook. Chicago, IL: Ibbotson Associates.

Milevsky, M.A. Forthcoming 2006. The Calculus of Retirement Income: Financial Models for Pensions and Insurance. *Cambridge University Press*.

Milevsky, M.A. and C. Robinson. 2005. “A Sustainable Spending Rate Without Simulation”, *Financial Analysts Journal*, vol. 64, no. 3 (Winter)

Milevsky, M.A. 2005. “The Implied Longevity Yield: A Note on Developing an Index for Payout Annuities”, *Journal of Risk and Insurance*, 2005, Vol. 72(2): 301-320

Milevsky, M.A. 2001. "Spending Your Retirement in Monte Carlo", *The Journal of Retirement Planning*, January/February: 21-29.

Pye, G. 2000. “Sustainable Investment Withdrawals.” *Journal of Portfolio Management*, vol. 26, no. 3 (Summer):13–27.

Whitehouse, Kaja. 2004. “Tool Tells How Long Nest Egg Will Last.” *Wall Street Journal* (31 August): Page D2

Table #1: Random Life & Returns: What Is Your Probability of Retirement Ruin?

Mortality Rate: **3.01%** ← ***Median Life (years):*** **23.00**

Retirement Income Spending Rate: **8.00%**

The Volatility of Portfolio's Investment Return:

Expected Return:	**5%**	**10%**	**15%**	**20%**	**25%**
1%	82.8%	84.2%	86.6%	89.7%	93.4%
3%	62.8%	66.5%	71.7%	77.5%	83.6%
5%	40.7%	46.4%	54.1%	62.5%	71.1%
7%	22.6%	28.6%	37.2%	47.1%	57.4%
10%	7.0%	11.0%	18.0%	27.3%	38.0%

Note: Under various portfolio returns $\{\mu\}$ and volatility $\{\sigma\}$ combinations, this chart displays the probability of retirement ruin – a.k.a. the retiree's RisQuotient – under a relatively high spending rate of $\{S = 8\%\}$ and a median remaining life of 23 years.

Table #2: Random Life & Returns: What Is Your Probability of Retirement Ruin?

Mortality Rate: **1.98%** ← ***Median Life (years):*** **35.00**

Retirement Income Spending Rate: **8.00%**

The Volatility of Portfolio's Investment Return:

Expected Return:	**5%**	**10%**	**15%**	**20%**	**25%**
1%	95.7%	95.4%	95.6%	96.7%	98.6%
3%	81.6%	82.8%	85.1%	88.3%	92.0%
5%	57.1%	61.9%	68.1%	74.8%	81.4%
7%	31.5%	39.0%	48.5%	58.4%	67.9%
10%	8.3%	14.1%	23.4%	34.6%	46.4%

Note: Under various portfolio returns $\{\mu\}$ and volatility $\{\sigma\}$ combinations, this chart displays the probability of retirement ruin – a.k.a. the retiree's RisQuotient – under a relatively high spending rate of $\{S = 8\%\}$ and a relatively long median remaining life of 35 years. Quite intuitively, the RisQuotient is high.

Table #3: Random Life & Returns: What Is Your Probability of Retirement Ruin?

Mortality Rate: **1.98%** ← ***Median Life (years):*** **35.00**

Retirement Income Spending Rate: **4.00%**

Expected Return:	***The Volatility of Portfolio's Investment Return:*** **5%**	**10%**	**15%**	**20%**	**25%**
1%	60.0%	67.0%	75.8%	85.2%	94.4%
3%	25.2%	34.8%	47.8%	61.8%	75.7%
5%	7.0%	13.3%	24.4%	38.7%	54.5%
7%	1.4%	3.9%	10.4%	21.3%	35.7%
10%	0.1%	0.4%	2.1%	7.1%	16.3%

Note: Under various portfolio returns $\{\mu\}$ and volatility $\{\sigma\}$ combinations, this chart displays the probability of retirement ruin – a.k.a. the retiree's RisQuotient – under a relatively low spending rate of $\{S = 4\%\}$ and a relatively long median remaining life of 35 years.

Table #4: Random Life & Returns: What Is Your Probability of Retirement Ruin?

Mortality Rate: **3.01%** ← ***Median Life (years):*** **23.00**

Retirement Income Spending Rate: **4.00%**

Expected Return:	***The Volatility of Portfolio's Investment Return:*** **5%**	**10%**	**15%**	**20%**	**25%**
1%	37.1%	44.8%	55.7%	67.9%	80.3%
3%	15.3%	21.9%	32.4%	45.7%	60.2%
5%	5.0%	8.8%	16.3%	27.5%	41.5%
7%	1.3%	3.0%	7.2%	15.1%	26.5%
10%	0.1%	0.5%	1.7%	5.2%	12.1%

Note: Under various portfolio returns $\{\mu\}$ and volatility $\{\sigma\}$ combinations, this chart displays the probability of retirement ruin – a.k.a. the retiree's RisQuotient – under a relatively low spending rate of $\{S = 4\%\}$ and a relatively low median remaining life of 23 years. Quite intuitively, the RisQuotients are lower compared to the previous three tables.

Table #5: BACK-OF-THE-ENVELOPE CALCULATION OF RETIREMENT RUIN RISK

RETIREMENT	Beta-adjusted Spending Level...															
ALPHA	**0.5**	**0.65**	**0.8**	**0.95**	**1.1**	**1.25**	**1.4**	**1.55**	**1.7**	**1.85**	**2**	**2.15**	**2.3**	**2.45**	**2.6**	**2.75**	**2.9**
4.50	0.1%	0.2%	0.4%	0.7%	1.2%	1.9%	2.8%	4.0%	5.4%	7.0%	8.9%	10.9%	13.2%	15.7%	18.3%	21.1%	24.0%
4.30	0.1%	0.2%	0.5%	1.0%	1.6%	2.5%	3.7%	5.1%	6.7%	8.6%	10.8%	13.1%	15.7%	18.4%	21.3%	24.3%	27.4%
4.10	0.1%	0.4%	0.8%	1.4%	2.2%	3.3%	4.7%	6.4%	8.4%	10.6%	13.0%	15.7%	18.5%	21.5%	24.6%	27.8%	31.1%
3.90	0.2%	0.5%	1.1%	1.9%	3.0%	4.4%	6.1%	8.1%	10.3%	12.9%	15.6%	18.6%	21.7%	24.9%	28.3%	31.6%	35.0%
3.70	0.3%	0.8%	1.5%	2.6%	4.0%	5.7%	7.7%	10.1%	12.7%	15.6%	18.6%	21.9%	25.2%	28.7%	32.2%	35.7%	39.2%
3.50	0.5%	1.2%	2.1%	3.5%	5.2%	7.3%	9.7%	12.4%	15.4%	18.6%	22.0%	25.5%	29.1%	32.8%	36.4%	40.1%	43.7%
3.30	0.8%	1.7%	3.0%	4.7%	6.8%	9.3%	12.1%	15.3%	18.6%	22.2%	25.8%	29.6%	33.4%	37.2%	41.0%	44.7%	48.3%
3.10	1.2%	2.4%	4.1%	6.2%	8.8%	11.7%	15.0%	18.5%	22.3%	26.1%	30.1%	34.0%	38.0%	41.9%	45.7%	49.4%	53.0%
2.90	1.8%	3.4%	5.5%	8.2%	11.3%	14.7%	18.4%	22.3%	26.4%	30.5%	34.7%	38.8%	42.9%	46.8%	50.6%	54.3%	57.8%
2.70	2.6%	4.7%	7.4%	10.6%	14.3%	18.2%	22.4%	26.7%	31.0%	35.4%	39.7%	43.9%	48.0%	51.9%	55.7%	59.3%	62.6%
2.50	3.7%	6.5%	9.9%	13.7%	17.9%	22.4%	26.9%	31.5%	36.1%	40.7%	45.1%	49.3%	53.3%	57.2%	60.8%	64.2%	67.4%
2.30	5.4%	8.9%	13.0%	17.5%	22.2%	27.1%	32.1%	36.9%	41.7%	46.3%	50.7%	54.8%	58.8%	62.5%	65.9%	69.1%	72.0%
2.10	7.6%	12.0%	16.9%	22.0%	27.3%	32.6%	37.8%	42.8%	47.6%	52.2%	56.5%	60.5%	64.2%	67.7%	70.8%	73.8%	76.4%
1.90	10.6%	16.0%	21.6%	27.4%	33.1%	38.7%	44.0%	49.1%	53.8%	58.2%	62.3%	66.1%	69.6%	72.7%	75.6%	78.2%	80.6%
1.70	14.7%	21.0%	27.3%	33.6%	39.6%	45.3%	50.7%	55.6%	60.2%	64.4%	68.2%	71.6%	74.8%	77.6%	80.1%	82.4%	84.4%
1.50	19.9%	27.1%	34.1%	40.7%	46.8%	52.5%	57.7%	62.4%	66.6%	70.4%	73.9%	76.9%	79.6%	82.1%	84.2%	86.1%	87.8%
1.30	26.5%	34.4%	41.8%	48.5%	54.5%	60.0%	64.8%	69.1%	72.9%	76.3%	79.2%	81.8%	84.1%	86.2%	87.9%	89.5%	90.8%
1.10	34.7%	43.1%	50.5%	56.9%	62.6%	67.6%	71.9%	75.6%	78.9%	81.7%	84.2%	86.3%	88.2%	89.8%	91.2%	92.4%	93.4%

Note: Instead of using the actual formula to compute the Retirement RisQuotient or ruin probability, the reader can look-up the number closest to the retirement alpha and beta-adjusted spending, in the above table.

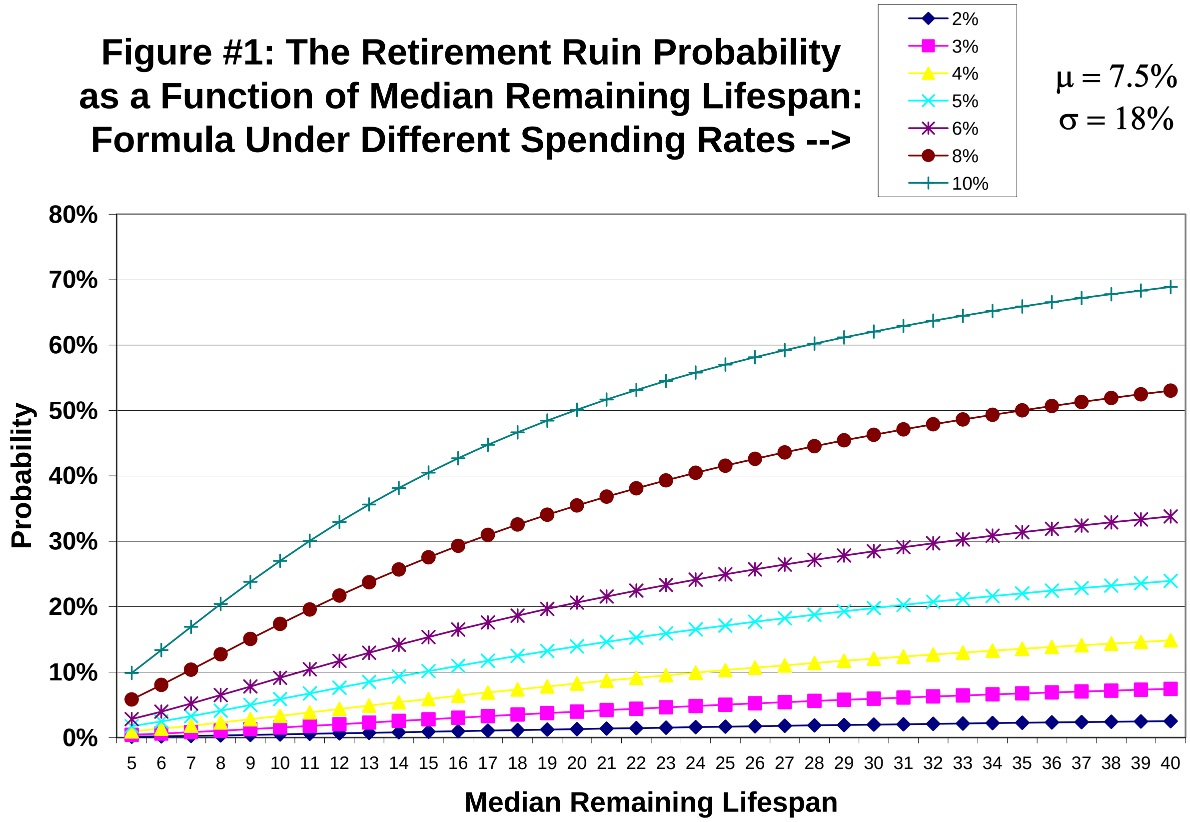
Table #6: You Retired and Are Planning on Spending 8% of Your Initial Nest Egg per Year Investment Portfolio Is Expected to Earn 7% with a Volatility of 20%

Retirement Age	Median Age at Death	Mortality Rate (%)	Ruin Probability with Life Annuities (%)	Ruin Probability without Life Annuities (%)
50.0	78.1	2.467	34.38	52.8
55.0	83.0	2.476	34.25	52.71
60.0	83.4	2.962	27.81	47.63
65.0	83.9	3.667	20.55	41.15
70.0	84.6	4.748	13.05	33.02
75.0	85.7	6.478	6.55	23.61
80.0	87.4	9.367	2.35	14.22

Note: These assume variable payout annuities with no management or insurance fees.

Note: Using part of your retirement portfolio to purchase variable life (a.k.a. payout or income) annuities will reduce the retirement ruin or RisQuotient. This is achieved by effectively increasing the portfolios return by the implied longevity yield or the mortality credits contained within the payout annuity.

Figure #1: The Retirement Ruin Probability as a Function of Median Remaining Lifespan: Formula Under Different Spending Rates -->



Note: The mortality rate λ – which is needed for the main formula -- is obtained by dividing the natural logarithm of 2, which is approximately 0.693 -- by the Median Remaining Lifespan (MRL). Thus for example when the MRL is 25 years, the relevant value of λ = 2.772%.