Stochastic Annuities Under Exponential Mortality

M.A. Milevsky[1] and J. Wang

York University and The IFID Centre

25 August 2004

[1]Milevsky is the contact author and can be reached at The Schulich School of Business, Finance Area, York University, 4700 Keele Street, Toronto, M3J 1P3, Canada or via Email at milevsky@yorku.ca. The authors would like to thank H. Huang and J. Dhaene for helpful discussions and conversations leading to this paper.

Abstract

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It is now well known amongst financial actuaries that the stochastic present value (SPV) of a *perpetuity* under a Brownian-driven force of interest is Reciprocal Gamma (RG) distributed. It is therefore tempting to suggest the same is true for *life annuities* under a constant force of mortality – i.e. when the future lifetime random variable is exponentially distributed – albeit with different parameter values, since the hazard rate can often be added to the interest rate. But alas, this result is not meant to be. In this brief note we prove that although the first and second moments of the exponential SPV – which match the RG distribution – lead to simple parameter shifts, the induced distribution of the SPV does not satisfy the required PDE. However, at the same time this paper demonstrates that the moment matched RG distribution based on the exponential lifetime is a remarkably simple and accurate approximation to the "true" SPV when the two densitites are calibrated via the median lifespan.

JEL Classification. G12, G22

1 Introduction and Motivation

It is now well known in the actuarial finance literature that the stochastic present value of a *perpetuity* under a Brownian-driven force of interest is Reciprocal Gamma distributed. This result can be traced back to work by Merton (1975) in finance, Majumdar and Radner (1991) in economics and Dufresne (1991) in actuarial science. It is therefore tempting to suggest the same result is true for *life annuities* under a constant force of mortality – i.e. when death is exponentially distributed – albeit with different parameter values. Indeed, in a number of other actuarial situations the value of a perpetuity $(1/r)$ at a force of interest r, can be transformed into an exponential life annuity $1/(r+\lambda)$ by adding the instantaneous force of mortality λ to the interest rate r. But alas, with stochastic present values (SPV) over an exponential horizon this convenient identity does not hold and the random value is definitely not Reciprocal Gamma distributed. In this brief note we provide a simple proof of this *negative result* based on moment matching techniques and the relationship between the distribution of the SPV and the PDE satisfied by the probability of ruin.

As a by-product of this "moment matching" approach we also derive an approximation for the distribution of the SPV. The approximation is based on replacing the drift μ and the variance σ^2 in the diffusion process by $\mu+2\lambda$ and $\sigma^2+\lambda$ respectively – where λ is the constant force of mortality – and then using the exact same formula as the perpetual SPV. Indeed, using a variety of numerical examples we demonstrate that the approximation is remarkably close to the correct values when compared against a Gompertz-Makeham distribution for the future lifetime. And, although the gap between the "true" and "approximate" probability values can be up to 5%, we beleive that the simplicity and ease-of-use for the approximation is worth this (slight) cost in accuracy.

2 Review of Known Results

We start with parameters $\{\mu, \sigma\}$, a one-dimensional Brownian motion $\mathbf{B}_s; s \geq 0$ and the geometric Brownian motion defined by:

$$\mathbf{Y}_t := e^{-(\mu-\sigma^2/2)s-\sigma\mathbf{B}_s}, \qquad Y_0 = 0. \tag{1}$$

We then construct the *generalized* stochastic present value (SPV) as the random variable

$$\mathbf{X_T} := \int_0^{\mathbf{T}} \mathbf{Y}_s ds := \int_0^{\infty} \mathbf{Y}_s 1_{\{\mathbf{T} \geq s\}} ds, \qquad X_0 = 0, \tag{2}$$

where, the second integral in equation (2) is the proper mathematical definition of the (less precise) first integral. The random variable $\mathbf{T}$ is defined by the instantaneous force of mortality (IFM) function $h(s)$, via:

$$\Pr\left[\mathbf{T} \leq t \mid h(0)\right] := 1 - e^{-\int_0^t h(s)ds}, \quad h(s) \geq 0. \tag{3}$$

The concept of an SPV in actuarial discounting as presented above was originally proposed in a paper by Buhlmann (1992) and the formal moments of the SPV $\mathbf{X}_t$ were first investigated by Beekman and Fuelling (1991).

With slight abuse of notation, we let $\mathbf{X}_\infty$ denote the SPV of a *perpetuity* and $\mathbf{X}_t$ denotes the SPV of a *term-certain annuity*. Examples of the IFM would include the Gompertz-Makeham law of mortality:

$$h(s) = h_0 + h_1 e^{h_2 s}, \qquad h(0) = h_0 + h_1, \tag{4}$$

which includes exponential mortality as a special case, and therefore:

$$\begin{aligned} \Pr\left[\mathbf{T} \leq t \mid h(0)\right] &= 1 - e^{-\int_0^t \left(h_0 + h_1 e^{h_2 s}\right) ds} \\ &= 1 - \left(e^{-h_0 t}\right)\left(e^{(-h_1/h_2)(1-e^{h_2 t}}\right). \end{aligned} \tag{5}$$

Figure #1 allow the reader to visualize the probability density function (PDF) of the SPV under various (increasing age) life expectancy assumptions.

Figure #1 Placed Here

The four curves "trace out" the PDF of the SPV for fixed risk and return parameter value ($\mu = 0.07$ and $\sigma = 0.20$), but under different parameterization for $\mathbf{T}$.

A number of authors – such as Milevsky (1997) or Norberg (1999) – have noted that if we construct a new stochastic process $\mathbf{W}_t$,

$$\mathbf{W}_t = \frac{1}{\mathbf{Y}_t}\left[w - \int_0^t \mathbf{Y}_t dt\right], \qquad W_0 = w \tag{6}$$

where $\mathbf{B}_t$ is the same Brownian motion used in equation (2), then $\mathbf{W}_t$ satisfies the stochastic differential equation,

$$d\mathbf{W}_t = (\mu \mathbf{W}_t - 1)dt + \sigma \mathbf{W}_t d\mathbf{B}_t \qquad W_0 = w. \tag{7}$$

See the classical book by Karatzas and Shreve (1992, Chapter 5) for demonstration based on Ito's formula. But, much more importantly,

$$\begin{aligned} \phi(t,w) &:= \Pr\left[\inf_{t\le s\le \mathbf{T}} \mathbf{W}_s \le 0 \mid W_t = w\right] \\ &= \Pr\left[\mathbf{W}_{\mathbf{T}} \le 0 \mid W_t = w\right] \\ &= 1 - \Pr\left[\mathbf{X}_{\mathbf{T}} \le w \mid h(t)\right]. \end{aligned} \tag{8}$$

In other words, the probability of ruin for the process $\mathbf{W}_t$ – conditional on an initial value w – is equal to one-minus the cumulative distribution function (CDF) of the SPV evaluated at the same w. The intuition behind this counter-inutive identity stems from the fact the $\mathbf{W}_t$ will only "cross" the value of zero once, and this is precisely when $\mathbf{X}_t$ is greater than w.

The intriguing relationship defined by equation (6) implies that the Partial Differential Equation (PDE) satisfied by the ruin probability evaluated at $W_0 = w$, must also be satisfied by (one minus) the probability that the SPV $\mathbf{X}_t$ is less than or equal to w. As mentioned earlier, this fact was exploited by Milevsky (1997) and Norberg (1999) to obtain the CDF of the perpetual SPV. Indeed, this idea leads to a complete alternative methodology for deriving the probability laws of any SPV, by focusing on the probability of ruin of the induced process.

We refer the interested reader to Huang, Milevsky and Wang (2004) for a formal proof that the ruin probability for the process defined by equation (6) satisfies the PDE given by:

$$\frac{\partial \phi(t,w)}{\partial t} + (\mu w - 1)\frac{\partial \phi(t,w)}{\partial w} + \frac{1}{2}\sigma^2 w^2 \frac{\partial^2 \phi(t,w)}{\partial w^2} - \phi(t,w)h(t) = 0, \tag{9}$$

with initial conditions $\phi(t,0) = 0, \phi(t,\infty) = 1$, where $h(t)$ is the instantaneous force of mortality defined above. The perpetuity case is specified by using $h(t) = 0$, which implies a time invariance $\partial\phi(t,w)/\partial t = 0$. We use the short-hand notation $\phi(w)$ for this special case. The solution to the PDE in equation (9), which now becomes an ODE, must satisfy:

$$\frac{\phi''(w)}{\phi'(w)} = \frac{1-(\alpha+1)w\beta}{w^2\beta}, \tag{10}$$

where $\alpha := 2\mu/\sigma^2 - 1$, $\beta := \sigma^2/2$. As mentioned earlier, the classical result is that:

$$\phi(w) = \Pr\left[\mathbf{X}_\infty \leq w\right] = \mathtt{G}(\alpha, \beta \mid 1/w) := \frac{\beta^{-\alpha}}{\Gamma(\alpha)} \int_0^{1/w} x^{(\alpha-1)} e^{-(x/\beta)} dx \tag{11}$$

where $\mathtt{G}(\alpha, \beta \mid \cdot)$ denotes the cumulative distribution function (CDF) of a Gamma random variable. It is also often stated that $\mathbf{X}_\infty$ itself is Reciprocal Gamma distributed with a CDF denoted and defined by,

$$\Pr\left[\mathbf{X}_\infty \leq w\right] = \mathtt{RG}(\alpha, \beta \mid w) := 1 - \frac{\beta^{-\alpha}}{\Gamma(\alpha)} \int_0^{w} x^{-(\alpha+1)} e^{-(1/x\beta)} dx. \tag{12}$$

The validity of this "Perpetuity Gamma" result can be confirmed by taking the ratio of the second to the first derivative of equation (12) with respect to w, and then substituting into equation (10).

The first two moments of the Reciprocal Gamma (RG) distribution are denoted generically by $M^{(1)}$ and $M^{(2)}$ and equal to:

$$M^{(1)} = \frac{1}{\beta(\alpha-1)}, \quad M^{(2)} = \frac{1}{\beta^2(\alpha-1)(\alpha-2)}. \tag{13}$$

This imposes a natural condition for the existence of the second moment, namely that $\alpha > 2$, which implies that $\mu > 3\sigma^2/2$. Note that the expectation of the *perpetuity* SPV will exist as long as $\mu > \sigma^2$, but the drift μ must be even larger for the SPV of the perpetuity to be RG distributed. We will return to the relevance of these conditions later in the analysis. For now it is important to note that equation (13) induces a one-to-one relationship between the parameters α, β and the moments M^1, M^2. Indeed, one can invert the moment equations and solve for the implied α, β parameters, which leads to:

$$\alpha = \frac{2M^{(2)} - M^{(1)}M^{(1)}}{M^{(2)} - M^{(1)}M^{(1)}}, \qquad \beta = \frac{M^{(2)} - M^{(1)}M^{(1)}}{M^{(2)}M^{(1)}}. \tag{14}$$

Thus, if we know the first two moments of the SPV we can invert them and "solve" for the α, β values listed above.

3 Main Proof by Contradiction

Under an exponential mortality assumption, the hazard rate is $h(t) := \lambda$ and the expected (or mean) future lifetime is equal to:

$$E[\mathbf{T}] = \int_0^\infty t\, \lambda e^{-\lambda t} dt = \frac{1}{\lambda}. \tag{15}$$

The *median future lifetime* is obtained via:

$$0.5 = e^{-\lambda\tau} \qquad \Longleftrightarrow \qquad \tau = \frac{\ln[2]}{\lambda} < \frac{1}{\lambda}. \tag{16}$$

For example, when $\lambda = 0.05$ the median future lifetime $\tau = \ln[2]/0.05 = 13.862$ years versus a life expectancy of $E[\mathbf{T}] = 20$ years, and when $\lambda = 0.1$ we get that $\tau = \ln[2]/0.1 = 6.931$ years, versus a life expectancy of $E[\mathbf{T}] = 10$ years. The distinction between median and mean lifetime will be used later. The probability of surviving to the mean lifetime of $t = 1/\lambda$ is $e^{-\lambda/\lambda} = e^{-1} = 36.78\%$.

Our motivating conjecture is that when $\mathbf{T}$ is exponentially distributed with IFM parameter $h(t) = \lambda$ that $\mathbf{X_T}$ is also Reciprocal Gamma distributed, albeit with (some unknown) parameters $\tilde{\alpha}$ and $\tilde{\beta}$ that depend on μ, σ as well as λ. Our proof that this is wrong, is by contradiction. If, indeed, $\mathbf{X_T}$ is Reciprocal Gamma distributed, then its moments, denoted by $M_\lambda^{(1)}$ and $M_\lambda^{(2)}$ will induce parameter values $\tilde{\alpha}$ and $\tilde{\beta}$ based on equation (14). We can then substitute the CDF of the (modified) RG into the PDE equation (9) to check if indeed it is satisfied. But in fact, it fails this test and hence it means that we only have an approximation and not a precise result.

We will use the following relationship, which is based on Fubini's theorem, a number of times in the paper.

$$E[\mathbf{X}_\mathbf{T}^n] = \int_0^\infty E[\mathbf{Y}_s^n] E[1_{\{\mathbf{T} \geq s\}}] ds = \int_0^\infty E[\mathbf{Y}_s^n] \left(1 - \Pr\left[\mathbf{T} < s\right]\right) ds \tag{17}$$

We start by defining the following intermediate variables $n_0 = \mu - \sigma^2/2$, $n_1 = \mu - \sigma^2$, $n_2 = \mu - 3\sigma^2/2$ and $n_3 = \mu - 2\sigma^2$, which implies that $n_0 \geq n_1 \geq n_2 \geq n_3$. In all future proofs and derivation we will operate under the restrictive case where $n_3 > 0$, which in turn implies that our drift μ is sufficiently larger than our volatility σ, which is required for convergence on the SPV integral.

Recall that using Fubini's theorem on equation (2), leads to an explicit expression for the moments of the SPV:

$$M_t^{(1)} := E[\mathbf{X}_t] = \int_0^t e^{-n_1 s} ds = \frac{1 - e^{-n_1 t}}{n_1} \tag{18}$$

and

$$
\begin{aligned}
M_t^{(2)} &:= E[\mathbf{X}_t^2] \\
&= \frac{2}{n_3}\int_0^t \left(e^{-n_1 s} - e^{-2n_2 s}\right) ds \\
&= \frac{2}{n_3}\left(\frac{1-e^{-n_1 t}}{n_1} - \frac{1-e^{-2n_2 t}}{2n_2}\right).
\end{aligned}
\tag{19}
$$

In the event that $t \to \infty$, the respective values converge to $M_\infty^{(1)} = (n_1)^{-1}$ and $M_\infty^{(2)} = (n_1 n_2)^{-1}$, or $M_\infty^{(1)} = (\mu - \sigma^2)^{-1}$ and $M_\infty^{(2)} = ((\mu-\sigma^2)(\mu - 3\sigma^2/2))^{-1}$, using the values of μ, σ.

Now, when $\Pr[\mathbf{T} > t] = \exp(-\lambda t)$, which is the exponential case, the moments are:

$$
M_\lambda^{(1)} := E[\mathbf{X}_\lambda] = \int_0^\infty e^{-(n_1+\lambda)s} ds = \frac{1}{n_1+\lambda} \tag{20}
$$

and

$$
\begin{aligned}
M_\lambda^{(2)} := E[\mathbf{X}_\lambda^2] &= \frac{2}{n_3}\int_0^\infty \left(e^{-(n_1+\lambda)s} - e^{-(2n_2+\lambda)s}\right) ds \\
&= \frac{2}{n_3}\left(\frac{1}{n_1+\lambda} - \frac{1}{2n_2+\lambda}\right) \\
&= \frac{2}{(n_1+\lambda)(2n_2+\lambda)},
\end{aligned}
\tag{21}
$$

since $2n_2 - n_1 = n_3$. Finally, by replacing μ, σ into the values of n_1 and n_2, we are left with:

$$
M_\lambda^{(1)} = \frac{1}{\mu + \lambda - \sigma^2} = \frac{1}{\tilde{\mu} - \tilde{\sigma}^2}, \tag{22}
$$

$$
M_\lambda^{(2)} = \frac{2}{(\mu+\lambda-\sigma^2)(2\mu - 3\sigma^2 + \lambda)} = \frac{2}{(\tilde{\mu}-\tilde{\sigma}^2)(2\tilde{\mu} - 3\tilde{\sigma}^2)}, \tag{23}
$$

where the modified drift and volatility variables are $\tilde{\mu} := \mu + 2\lambda$ and $\tilde{\sigma}^2 := \sigma^2 + \lambda$, respectively. Thus, the first and second moments of the SPV under exponential mortality are given.

Now, if indeed the SPV is RG, since the first two moments are known, they must match the moments of RG. This would imply that:

$$
\Pr\left[\mathbf{X}_\lambda \le w\right] = \mathtt{RG}(\tilde{\alpha}, \tilde{\beta} \mid w) := 1 - \frac{\tilde{\beta}^{-\tilde{\alpha}}}{\Gamma(\tilde{\alpha})}\int_0^w x^{-(\tilde{\alpha}+1)} e^{-(1/x\tilde{\beta})} dx, \tag{24}
$$

but with where $\alpha = 2\tilde{\mu}/\tilde{\sigma}^2 - 1$, $\beta = \tilde{\sigma}^2/2$. We pause for a moment to let this statement sink in, since it the basis of the approximation that we later advocate and use in section 4. The first and second moment of the SPV under exponential mortality can be precisely matched to the first two moments of the RG distribution by simply adding 2λ to the drift μ and λ

to the variance σ^2, and using the exact same formula as the perpetuity. Unfortunately, the excitement ends here since $\mathtt{RG}(\tilde{\alpha}, \tilde{\beta} \mid w)$ does not satisfy the fundamental PDE and:

$$(\mu w - 1)\mathtt{RG}_w + \frac{1}{2}\sigma^2 w^2 \mathtt{RG}_{ww} - \lambda \mathtt{RG} \neq 0, \tag{25}$$

and hence the exponential SPV is not Reciprocal Gamma distributed. **QED.**

3.1 Yet Another Perspective

Another way to attack this problem is to examine what happens when $\sigma \to 0$, and the SPV defined by equation (2) contains no Brownian component. In this case $\alpha = 2\mu/\lambda + 3$ and $\beta = \lambda/2$. The expected value of the SPV is $(\mu + \lambda)^{-1}$. In the current section we use alternative techniques to demonstrate that the ruin probability – and by analogy the CDF of the SPV under exponential future lifetime – satisfies:

$$\lim_{\sigma \to 0} \Pr\left[\mathbf{X_T} \leq \frac{1}{\mu + \lambda}\right] = \lim_{\sigma \to 0} \phi(t, \frac{1}{\mu + \lambda}) = 1 - \left(1 + \frac{\mu}{\lambda}\right)^{-\lambda/\mu}, \tag{26}$$

which collapses to $1/2$ when $\mu = \lambda$. This relationship may seem odd at first, but should make sense in a bit.

Let us use the non-bolded notation W_t to denote the non-Brownian version of equation (6) and (7) when $\sigma = 0$. The W_t process will now obey the ordinary differential equation (ODE),

$$dW_t = (\mu W_t - 1)\, dt, \qquad W_0 = w, \quad W_t \geq 0, \tag{27}$$

which – without any loss of generality – we define up to the point of ruin $W_{t^*} = 0$. The solution to the ODE is:

$$W_t = \begin{cases} \left(w - \frac{1}{\mu}\right) e^{\mu t} + \frac{1}{\mu} & t < t^* \\ 0 & t \geq t^* \end{cases}, \tag{28}$$

where t^* is the time of ruin. This value can be obtained exactly by solving:

$$(w - \frac{1}{\mu})e^{\mu t} + \frac{1}{\mu} = 0 \qquad \Longleftrightarrow \qquad t^* = \frac{1}{\mu} \ln\left[(w\mu - 1)^{-1}\right] \tag{29}$$

Finally, when the initial value of the function/process W_0 is arbitrarily set equal to $w = (\lambda + \mu)^{-1}$, the ruin time t^* can be simplified to:

$$t^* = \frac{1}{\mu} \ln[1 + \frac{\mu}{\lambda}] \tag{30}$$

And, when $\mu = \lambda$ the value of $t^* = \ln[2]/\lambda$, which is exactly the *median* life span. In the limit, as $\mu \to 0$, the ruin time is precisely the life expectancy $1/\lambda$, since

$$\lim_{\mu \to 0} \frac{1}{\mu} \ln[1 + \frac{\mu}{\lambda}] = \frac{1}{\lambda} \tag{31}$$

Finally, the probability of not surviving to the point at which W_t hits zero, is:

$$\Pr\left[\inf_{0 \le s \le \mathbf{T}} W_s > 0\right] := 1 - e^{-\lambda t^*} = 1 - e^{-\frac{\lambda}{\mu} \ln[1+\frac{\mu}{\lambda}]}. \tag{32}$$

Yet another "proof" that the SPV is not RG distributed is to substitute $\sigma = 0$ into equation (24) and realize that it does not lead to equation (32).

4 How Good is the Exponential Approximation?

There are quite a number of numerical procedures that are used to approximate the SPV in equation (2). One very successful method is based on comonotonic approximations and is pursued in a series of papers by Dhaene, Denuit, Goovaerts, Kaas and Vyncke (2002). Another approach is based on Laplace transforms and is pursued by Vanneste, Goovaerts and De Schepper and Dhaene (1997). In this paper we take a moment-matching approach used by Milevsky and Robinson (2000) or Huang, Milevsky and Wang (2004), but use the exponential distribution calibrated to *median* lifetime values leading to equation (24).

We start with a given Gompertz-Makeham parametrization h_0, h_1, h_2 for the hazard rate. We select parameters μ, σ and we compute the CDF for the SPV (i.e. one minus the ruin probability) for various initial levels of w using numerical precise methods built around the PDE in equation (9). We also approximate the random variable $\mathbf{T}$ with an exponential distribution driven by λ, for which the median lifespans are matched. This "median matching" induces a value of λ which we then use to obtain CDF values and compare with the "true" Gompertz-Makham assumption.

In the numerical examples that follow, we "picked" h_0, h_1, h_2 values that best fit the RP2000 (Unisex) pension mortality table and then "solved" for the λ value that would lead to the same exact median lifespan. We further assumed that $\mu = 0.07$ and $\sigma = 0.20$ which is consistent with capital market history according to Ibbotson Associates (2002).

TABLE #1 Placed Here

For example, according to the RP2000 (Unisex) table, the median lifespan for a 50-year-old is 27 years, i.e. the probability of survival ${}_{27}p_{50} = 50\%$. Thus, if we approximate the future-lifetime random variable T using an exponential distribution with the same median, the constant instantaneous force of mortality parameter (IFM) is: $\lambda = 0.0257 = \ln[2]/27$. Now, we can also approximate the RP2000 (Unisex) survival probabilities from age 50 with a (more accurate) Gompertz-Makeham IFM curve which leads to h_0, h_1 and h_2 values using an error minimization algorithm. We then compute the probability $\Pr[X_T < w]$ for a variety of $\{w = 50, 20, 10...\}$ values using the exact algorithm from the PDE in equation (9) and the approximate algorithm in equation (24). The results are as follows. For $w = 50$, the "exact" CDF value is 3.8% and the "approximate" CDF value is 4.1%. The gap is 0.3%. Likewise, for $w = 10$, the "exact" CDF value is 61.8% and the "approximate" CDF value is 65.8%. The gap is 4.0%.

In fact, using a Kolmogorov-Smirinov test for the goodness of fit between these two candidate distributions, we were unable to reject the Null hypothesis that they are the same – although we know they are not – at the 5% confidence level. Moreover, after extensive testing we find that the gap between the two distributions, i.e. the numerical solution to the PDE and the analytic RG approximation, never exceeds 5% when the risk/return parameters μ, σ and the value of w are in an economically feasible ranges. Thus, although we can certainly locate pathological values where the gap between the two densities is quite large, we find these cases to be economically uninteresting. Recall that one of the main situations in which such an approximate algorithm would be of interest is in the context of retirement planning and sustainable withdrawal rates, which have been explored extensively in a number of recent papers such as Albrecht and Maurer (2002) or Blake, Cairns, and Dowd (2003).

4.1 Why Does it Work?

We suspect that this approximation does a "good job" of replicating the true solution to equation (9) for a variety of reasons. First, we know for certain that when $\lambda \to 0$ (i.e. the discounting is over very long periods) the solution is exact and thus it is not surprising it is also the case for λ values that are not "too" large. Second, when we think of this problem from the dual *probability of ruin* perspective, as exemplified in equation (8), we can use well-established intuition from ruin theory. Namely, when a stochastic process gets ruined, it either does so very early in time, or it never gets ruined since the process escapes the "danger zone". Thus, if we make sure to calibrate the two random lifetime densities so that the 50% mark is the same, we likely catch the regions in which ruin is around the median lifespan. And, the values in time far to the left or right of this calibration point – for which the two densities do not match well – become less relevant.

5 Conclusion

It might seem odd to devote a paper to proving that a (quite remarkable) specific result does not apply under more general conditions, but we believe the technique we used is more valuable than the negative result itself. Namely, the duality between the stochastic present value and the ruin probability leads to a PDE representation for the SPV that can always be used to verify "candidate" densities for the CDF. Moreover – and on a practical level – we have also demonstrated that a suitably calibrated exponential approximation to the SPV can lead to CDF values that are remarkably close to the correct values when the constant force of mortality is chosen to match the median future lifetime random variable.

In sum, we are advocating that the CDF of the SPV defined by $\mathbf{X_T}$ be approximated by:

$$\mathtt{RG}(\frac{2\mu + 4\lambda}{\sigma^2 + \lambda}, \frac{\sigma^2 + \lambda}{2} \mid w),$$

where RG is the Reciprocal Gamma distribution, μ and σ are the drift and diffusion coefficient of the return generating process and λ is the instantaneous force of mortality which leads to the same median lifespan as the "true" variable $\mathbf{T}$.

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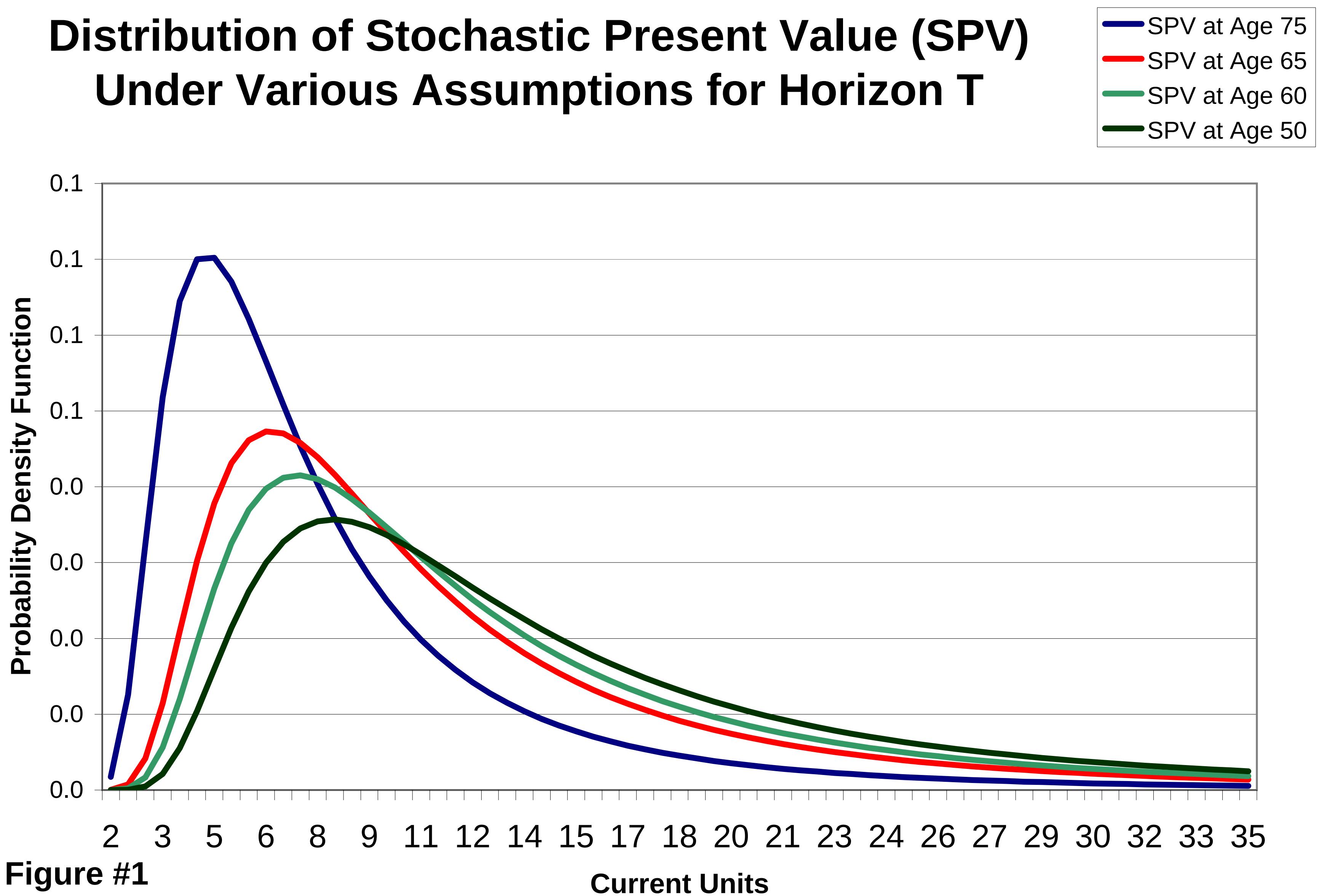
Distribution of Stochastic Present Value (SPV)
Under Various Assumptions for Horizon T
SPV at Age 75
SPV at Age 65
SPV at Age 60
SPV at Age 50
Probability Density Function
0.1
0.1
0.1
0.1
0.0
0.0
0.0
0.0
0.0
2 3 5 6 8 9 11 12 14 15 17 18 20 21 23 24 26 27 29 30 32 33 35
Current Units
Figure #1

TABLE #1: The probability the SPV of $1 -- until random time T -- is greater than…

Current Age	Median Age-at-Death	λ Implied	w	$ 50.00	$ 33.33	$ 25.00	$ 20.00	$ 16.67	$ 14.29	$ 12.50	$ 11.11	$ 10.00
50	77	0.0257	Approx.	4.1%	9.9%	17.4%	26.0%	34.9%	43.5%	51.7%	59.2%	65.8%
			Exact.	3.8%	10.0%	18.1%	26.9%	35.5%	43.5%	50.6%	56.7%	61.8%
			Diff:	0.3%	-0.2%	-0.7%	-0.9%	-0.6%	0.1%	1.1%	2.4%	4.0%
55	82	0.0257	Approx.	4.1%	9.9%	17.4%	26.0%	34.9%	43.5%	51.7%	59.2%	65.8%
			Exact.	3.5%	9.5%	17.4%	26.2%	34.9%	43.0%	50.2%	56.4%	61.6%
			Diff:	0.6%	0.4%	0.0%	-0.2%	0.0%	0.6%	1.5%	2.7%	4.2%
60	83	0.0301	Approx.	3.4%	8.4%	15.1%	22.8%	31.0%	39.2%	47.1%	54.5%	61.2%
			Exact.	2.6%	7.7%	14.9%	23.0%	31.2%	39.1%	46.2%	52.5%	57.8%
			Diff:	0.8%	0.6%	0.2%	-0.2%	-0.2%	0.1%	0.9%	2.0%	3.4%
65	84	0.0365	Approx.	2.7%	6.7%	12.3%	19.0%	26.3%	33.9%	41.3%	48.4%	55.1%
			Exact.	1.8%	5.9%	11.9%	19.2%	26.7%	34.2%	41.2%	47.4%	52.8%
			Diff:	0.9%	0.9%	0.4%	-0.1%	-0.4%	-0.4%	0.1%	1.0%	2.3%
70	85	0.0462	Approx.	1.9%	4.9%	9.3%	14.7%	20.7%	27.2%	33.9%	40.4%	46.8%
			Exact.	1.1%	3.9%	8.6%	14.7%	21.3%	28.2%	34.8%	40.8%	46.1%
			Diff:	0.8%	1.0%	0.7%	0.0%	-0.6%	-0.9%	-0.9%	-0.4%	0.6%
75	87	0.0578	Approx.	1.3%	3.5%	6.8%	11.0%	15.9%	21.3%	27.0%	32.8%	38.6%
			Exact.	0.5%	2.3%	5.7%	10.4%	16.1%	22.2%	28.2%	34.0%	39.3%
			Diff:	0.8%	1.2%	1.1%	0.6%	-0.2%	-0.9%	-1.3%	-1.2%	-0.7%
80	89	0.0770	Approx.	0.8%	2.1%	4.3%	7.1%	10.6%	14.6%	18.9%	23.5%	28.3%
			Exact.	0.2%	0.9%	2.8%	5.9%	10.1%	14.9%	20.1%	25.3%	30.2%
			Diff:	0.6%	1.2%	1.5%	1.2%	0.5%	-0.4%	-1.2%	-1.8%	-1.9%

Return:	7.0%
Volatility:	20.0%

Notes: *We assume the Unisex version of the (non-projected) RP2000 Mortality Table from the Society of Actuaries*